Updating Variational (Bewley) Preferences

José Heleno Faro
Ana Santos

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José Heleno Faro[a], Ana Santos[a]

[a]Insper, Rua Quatá 300, Vila Olímpia 04546-042 São Paulo, Brazil

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Abstract

This paper investigates the dynamic consistent updates of incomplete preferences relations in the class of variational Bewley preferences (VBP). We show that this conditional preferences are also VBP that reveals the same ranking over consequences as the unconditional relation and are represented by an ambiguity index obtained through the full Bayesian update of the *ex ante* ambiguity index.

Moreover, we study *ex post* forced choice relations, which captures choices that must be taken by a decision maker after learning some relevant event. Formally, they are monotone continuous weak orders. We show that any *ex post* forced choice relation that preserves the ambiguity attitudes of a given *ex ante* VBP has a variational functional representation with the same raking over consequences and the ambiguity index of the corresponding dynamic consistent update. Thus, our result can be viewed as a novel foundation for the full Bayesian update for the class of variational preferences.

Keywords: Dynamic consistency, full Bayesian update, forced choices, incomplete preferences, variational Bewley preferences, variational preferences.

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1 Introduction

One of the most fundamental issue in economics consists in understand how individuals make decisions in situations that involves uncertainty when probabilities do not objec-

* *Contacts*: *josehf@insper.edu.br*, *anactros1@insper.edu.br*. We wish to thank the participants of seminars at Insper, Université Paris I Panthéon-Sorbonne, Bocconi University and conferences SAET 2017 and FUR 2018 for their helpful comments. Santos gratefully acknowledges financial support from Fapesp (Grant No. 2017/09955-4 and 2018/00215-0). Faro is grateful for financial support from the CNPq-Brazil (Grant No. 308183/2019-3).

tively exist.[1] The belief theories of Ramsey (1931), de Finetti (1937) and Savage (1954) provide the axiomatic foundations for the widely known subjective expected utility (SEU) theory, where subjective probabilities are derived from preferences. Later, Anscombe and Aumann (1963) provide a simpler characterization of SEU in a framework of acts mapping states to lotteries, which allows to describe the axioms in a similar way of those proposed by von Neumann and Morgenstern (1944). The SEU approach is the main reference for the use Bayesian models in economics (e.g., general equilibrium models with heterogeneous beliefs and games with incomplete information). However, there are behavior patterns under subjective uncertainty that cannot be quantified by the Bayesian paradigm. This is suggested by the ambiguity averse literature that follows from the works of Knight (1921) and Ellsberg (1961).[2]

Moreover, one necessary condition for applying the SEU model is that the decision maker (DM henceforth) must be able to pair-wise compare all available choice options. The Knightian decision theory proposed by Bewley (1987, 2002) departure from this assumption and obtains an unanimity rule with multiple priors. This model was generalized by Faro (2015) with a variational weighted unanimity rule. The class of variational Bewley preferences (VBP) captures the idea of "indecisiveness" due to informational and/or cognitive limitations the DM recognizes.[3] Despite the difficulty of testing for the completeness axiom, Cettolin and Riedl (2019) and Costa-Gomes *et al.* (2019) conduct laboratory experiments that find behavioral patterns consistent with the existence of incomplete preferences.[4]

Most problems in economics are sequential and understand how individuals update their choices is essential for applied situations. In special, the well-know dynamic consistency property i.e., the DM carry out plans made *ex ante*)[5] introduced by Savage (1954, p. 22) guarantees that usual dynamic programming methods can be applied.[6] In the framework of expected utility dynamic consistency is equivalent to the Baye's rule, which is widely applied. However, in situations that involves ambiguity there is not an unique

[1]The main reference for the case where probabilities are objectively given is the axiomatic foundation of expected utility under risk of von Neumann-Morgenstern (1944).

[2]For instance, the counterexamples suggested by Ellsberg (1961) and referred to as "Ellsberg's paradoxes" in the literature have been confirmed by many empirically studies (Camerer and Weber 1992, Machina and Siniscalchi 2014).

[3]Other authors that discusses incompleteness due to "indecisiveness" are Eliaz and Ok (2006), Gilboa (2009), Gilboa *et al.* (2009) and Minardi and Savochkin (2015). For incompleteness in tastes see Ok, Ortoleva, and Riella (2012) and Galaabaatar and Karni (2013).

[4]Another possible explanation for Cettolin and Riedl (2019) results is the presence of preference for randomization and for Costa-Gomes *et al.* (2019), the presence of menu-dependent priors. Others empirical works that investigate this issue are Sautua (2017) and Danan and Ziegelmeyer (2006).

[5]See Ghirardato (2002).

[6]See Stokey, Lucas and Prescott (1989).

answer and some of them violates dynamic consistency.[7]

In this paper, first we study the *dynamic consistent update* of incomplete preferences relations in the class of VBP. Given an event that captures a partial resolution of uncertainty that is relevant to the DM, we show that the conditional relation belongs to the same class of preferences, preserves the raking over consequences, and the ambiguity index follows the full Bayesian update. Second, motivated by the fact that in many real situations the DM must compare any pair of acts, we consider forced choice relations as monotone continuous weak orders derived from an *ex ante* VBP. In addition we impose some condition on the attitude towards uncertainty of those relations.

For this purpose, in a similar way as in Gilboa *et al.* (2010), we consider a DM characterized by two binary relations.[8] However, instead of their static framework we consider one with partial resolution of uncertainty, summarized by an event E. The idea is that initially the DM has an *ex ante* preference relation over acts that is potentially both intransitive and incomplete. Then, after receiving some information he is forced to exhibit a rank among any pair of acts. The DM must revise all potential inconsistencies (e.g., intransitive patterns of choice) of the unconditional preference. Most importantly, we want to find the conditional relation that exhibit the same the ambiguity attitude of the unconditional one, after learning E. Normatively, forced choice relations are driven by rational principles and should exhibit the same ambiguity attitude of the unconditional relation.

Inspired by Ghirardato and Marinacci (2002) that deals with complete preferences, we adapt their notion of comparative ambiguity attitude to the context of incomplete preferences with partial resolution of uncertainty, incorporating the concerns of Faro, Ok and Riella (2018). Our main result shows that any *ex post* forced choice relation derived from an *ex ante* VBP has a variational representation of Maccheroni, Marinacci and Rustichini (2006) (MMR henceforth) characterized by the same affine utility index over consequences and the ambiguity index of the corresponding dynamic consistent update.

The rest of the paper is organized as follows. Section 2 introduces the setup, the notation and the relevant mathematics. Section 3 presents our results about the updating rule for the class of VBP. Section 4 characterizes the comparative ambiguity attitude that we need and then presents our main result about finding and *ex post* forced choice relation from an unconditional VPB. In section 5 we discuss the connections between our results and the works of Faro (2015), Li (2015), Faro and Lefort (2019) and Gilboa *et al.* (2010). Finally, Section 6 concludes and proof are collected in Appendix A.

[7] Gilboa and Schmeidler (1993) and Hanany and Klibanoff (2007) also studied the problem of updating in the context of ambiguity.

[8] Other papers studying pairs of binary relations are Kopylov (2009), Cerreia-Vioglio (2016), Faro (2015), Faro and Lefort (2019) and Cerreia-Vioglio *et al.* (2020).

2 Framework

Consider a set S of *states of nature (the world)*, endowed with a σ-algebra Σ of subsets called *events*, and a non-empty set X of *consequences*. We say that a function $f : S \to X$ is simple if $f(S) := \{f(s) : s \in S\}$ is a finite set. A simple function $f : S \to X$ is Σ-measurable if $\{s \in S : f(s) = x\} \in \Sigma$ for all $x \in X$. We denote by $\mathcal{F}$ the set of all simple and Σ-measurable functions. We call a mapping $f \in \mathcal{F}$ by an *act*. Moreover, we denote by $B_0(\Sigma)$ the set of all simple real-valued Σ-measurable functions $a : S \to \mathbb{R}$. The norm in $B_0(\Sigma)$ is given by $\|a\|_\infty = \sup_{s\in S} |a(s)|$ (called the *sup norm*) and $B(\Sigma)$ will denote the supnorm closure of $B_0(\Sigma)$. In another way, $B_0(\Sigma)$ is the vector generated by the indicator functions of the elements of Σ, endowed with the supnorm (for more details, see Dunford and Schwarts (1988), Section 5 of Chapter IV). We denote by $ba(\Sigma)$ the Banach space of all finitely additive set functions on Σ endowed with the total variation norm. It is isometrically isomorphic to the norm dual of $B_0(\Sigma)$. Note also that the weak* topology $\sigma(ba, B_0)$ of $ba(\Sigma)$ coincides with the eventwise convergence topology. Throughout the paper, we assume that any subset of $ba(\Sigma)$ is endowed with the topology inherited from the weak* topology.

Given a mapping $u : X \to \mathbb{R}$, the function $u(f) : S \to \mathbb{R}$ is defined by $u(f)(s) = u(f(s))$, for all $s \in S$. We note that $u(f) \in B_0(\Sigma)$ whenever f belongs to $\mathcal{F}$. Let x be a consequence in X, abusing notation we define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. Hence, we can identify X with the set of constant acts in $\mathcal{F}$. Additionally, we assume that set of consequences X is a convex subset of a vector space. For instance, this is the case if X is the set of all simple lotteries on a set of *outcomes* Z. In fact, it is the classic setting of Anscombe and Aumann (1963) as re-stated by Fishburn (1970). Using the linear structure of X, we can define as usual for every $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$ the act $\alpha f + (1-\alpha)g : S \to X$ by $(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s)$, $\forall s \in S$. Also, given two acts $f, g \in \mathcal{F}$ and an event $E \in \Sigma$, we denote by fEg the act delivering the consequences $f(s)$ in E and $g(s)$ in $E^c := S \backslash E$ (the complement of E).

We denote by $\Delta(S, \Sigma) := \Delta$ the set of all (finitely additive) probability measures $p : \Sigma \to [0, 1]$. Given an act $f \in \mathcal{F}$, a utility index u, and a probability measure $p \in \Delta$, the expected utility of f is denoted by $\int u(f)\, dp$.

We say that a mapping $\eta : \Delta \to [0, \infty]$ is grounded if $\{\eta = 0\} := \{p \in \Delta : \eta(p) = 0\} \neq \emptyset$. The effective domain of $\eta : \Delta \to [0, \infty]$ is defined by $\{\eta < \infty\} := \{p \in \Delta : \eta(p) < +\infty\}$ and $\overline{\{\eta < \infty\}}$ denotes its closure. Also, η is lower semi-continuous if $\{\eta \leq r\}$ is closed for each $r \geq 0$. Moreover, we denote by Δ^σ the set of all countably additive probabilities in Δ. In particular, given $q \in \Delta^\sigma$, we denote by $\Delta^\sigma(q)$ the set of all probabilities in Δ^σ that are absolutely continuous w.r.t. q, i.e., $\Delta^\sigma(q) = \{p \in \Delta^\sigma : p \gg q\}$, where $p \gg q$ means that $\forall A \in \Sigma$, if $q(A) = 0$, then $p(A) = 0$.

Functions of the form $\eta : \Delta \to [0, \infty]$ capture the degree of plausibility of each prior, for a DM that can be represented as in the present paper. We denote by $\mathcal{N}(\Delta)$ the class of those functions such that η is grounded, convex and lower semi-continuous. A mapping η is called an ambiguity index.

Given a binary relation $\succsim$ on $\mathcal{F}$, also called a preference relation, the symmetric and asymmetric parts of $\succsim$ are denoted by $\sim$ and $\succ$, respectively. For each event $E \in \Sigma$, the *ex-ante* preference relation $\succsim$ is associated to the *ex-post* preference relation $\succsim_E$ on $\mathcal{F}$, which is the conditional preference that emerges after learning that the event E is obtained (if $E = S$, we have $\succsim_S \equiv \succsim$). We say that an event E is *relevant* (w.r.t. $\succsim$) provided that, if there exist two consequences such that $x \succ y$, then $xEy \succ y$ and $x \succ yEx$. We denote by $\mathcal{R}_{\succsim}$ the family of all relevant events (w.r.t. $\succsim$).

Let p be a probability measure in Δ and $E \in \Sigma$ such that $p(E) > 0$. We denote by $p(\cdot|E)$ its Bayesian updating or its conditional probability (w.r.t. E), which is defined by

$$p(F|E) = \frac{p(E \cap F)}{p(E)}, \forall F \in \Sigma.$$

The corresponding conditional expected utility of an act $f \in \mathcal{F}$ is given by

$$\int_S u(f)\, dp(\cdot|E) = \frac{1}{p(E)} \int_E u(f)\, dp.$$

We extend the definition of Bayesian update to the ambiguity index as follows. Let η be an ambiguity index and for all $p_E \in \Delta(E)$, the function η_E is defined by a generalization of the full Bayesian update:

$$\eta_E(p_E) = \inf_{\{p \in \{\eta < \infty\} : p(\cdot|E) = p_E\}} \frac{\eta(p)}{p(E)}.$$

The idea is that taking the infimum over all probabilities with posterior p_E controls for any concern for model misspecification outside event E, which is irrelevant to $\succsim_E$ due to consequentialism.[9] Normalization by $1/p(E)$ captures a maximum likelihood intuition: probabilities p assigning a higher probability on the event that occurred are more likely to be selected and determine η_E.

Note that the standard case of full Bayesian update is captured by an ambiguity index given by η where there exists a nonempty, convex, and $\sigma(ba, B_0)$-compact set $\mathcal{C} = \{\eta < \infty\}$ with $\eta(\mathcal{C}) = 0$. In this case, we have that $\mathcal{C}^E := \{p(\cdot|E) : p \in \mathcal{C}\} = \{\eta_E < \infty\}$ with $\eta_E(\mathcal{C}^E) = 0$.

3 Updating Variational Bewley Preferences

The class of VBP is a model of choice under uncertainty that generalizes the representation proposed by Ghirardato, Maccheroni and Marinacci (2004) for Bewley (1987, 2002)

[9]Faro and Lefort (2019) show that consequentialism is implied by dynamic consistency in the context of Anscombe and Aumann acts (Proposition 2, p.10), which is our case.

incomplete preferences. The axiomatic foundation was proposed by Faro (2015) and the main feature of the VBP is that it may fail not only independence, but also completeness and transitivity. The DM's preferences are given by a binary relation $\succsim$ on $\mathcal{F}$ and the axioms that characterizes the class of preferences are as follows.

Axiom 1. Reflexivity*: For all* $f \in \mathcal{F}$, $f \succsim f$.

Axiom 2. Unambiguous Transitivity*: Suppose* $f \succsim g$*. If* $h(s) \succsim f(s)$ *for all* s*, then* $h \succsim g$*. Also, if* $g(s) \succsim h(s)$ *for all* $s \in S$*, then* $f \succsim h$.

Axiom 3. C-Completeness*: For all constant acts* x *and* y, $x \succsim y$ *or* $y \succsim x$.

Axiom 4. Continuity*: For all* $f, g, h \in \mathcal{F}$ *the sets:* $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succsim h\}$ *and* $\{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha)g\}$ *are closed in* $[0,1]$.

Axiom 5. Dominance Independence*: For all* $f, g, h_1, h_2 \in \mathcal{F}$*, and all* $\alpha \in (0,1)$*, if* $f \succsim g$ *and* $h_1 \succsim h_2$ *then* $\alpha f + (1-\alpha)h_1 \succsim \alpha g + (1-\alpha)h_2$.

Axiom 6. Unboundedness*: There are* $x, y \in X$ *such that for each* $\alpha \in (0,1)$*, there exist* $z, \hat{z} \in X$ *such that* $\alpha z + (1-\alpha)y \succ x \succ y \succ \alpha \hat{z} + (1-\alpha)x$.

Axioms 1, 4 and 6 are technical requirements that are standard in this literature. Axiom 2 relaxes the usual transitivity condition.[10] Transitivity must hold only when an act dominates the other in all states of nature. Axiom 3 captures the idea that incompleteness is due to uncertainty but not to tastes.[11] Axiom 5 requires some sort of independence on preferences.[12] It says that mixtures of preferred acts should be also a preferred act when compared to the same mixture of the dominated ones. Now we enunciate the main result of Faro (2015).

Theorem 7. *(*Theorem 1 of Faro (2015); p. 706*) Let* $\succsim$ *be a preference relation of the set of Anscombe and Aumann acts* $\mathcal{F}$*. Then the following conditions are equivalent:*

(i) $\succsim$ *satisfies axioms 1 to 6.*

(ii) There exists an affine utility index $u : X \to \mathbb{R}$*, with* $u(X) = \mathbb{R}$*, and a function* $\eta : \Delta \to [0, \infty]$ *that belongs to* $\mathcal{N}(\Delta)$ *such that for all* f *and* g *in* $\mathcal{F}$,

$$f \succsim g \Leftrightarrow \int u(f)dp + \eta(p) \geq \int u(g)dp, \text{ for all } p \in \Delta.$$

[10] Recall that: $\succsim$ is transitive if for all $f, g, h \in \mathcal{F}$, $f \succsim g$ and $g \succsim h$ imply $f \succsim h$.

[11] For a more general treatment of incompleteness, including also tastes, see Aumann (1962), Kannai (1963), Richter (1966), Peleg (1970), Ok (2002), Dubra, Maccheroni and Ok (2004), Ok, Ortoleva and Riella (2012).

[12] It worth mention that the classical independence axiom (i.e., for all $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1)$, $f \succsim g \Leftrightarrow \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$) and dominance independence are equivalent when transitivity holds. For a more detailed discussion about the relation between Independence and the VBP see Bastianello, Faro and Teles (2020).

For each u there is a unique $\eta^* : \Delta \to [0,\infty]$ consistent with our representation given by

$$\eta^*(p) = \sup\left\{\int (u(g) - u(f))dp\text{: } f \succsim g\right\}, \text{ for all } p \in \Delta.$$

As mentioned above, this representation relaxes the strictness of Bewley's unanimity rule by allowing the DM to have different weights for each prior, given by the function $\eta : \Delta \to [0,\infty]$.[13] Higher values of $\eta(p)$ indicate priors that have less weight and higher must be the difference between the expected utility of f and that of g to justify the choice of the former over the latter. In the special case in which some priors have full plausibility (i.e., $\eta(p) = 0$) and others are discarded (i.e., $\eta(p) = \infty$), we recover the Bewley representation.[14]

Hence, the ambiguity index determines a weighted unanimity rule: for an act to be preferred to other all the priors with higher weight must agree about the overall decision. That is, when considering p, the DM accepts to lose a fixed amount in terms of net expected utility of the acts given by $\eta(p)$. We may interpret $\eta^*(p)$ as the maximum expected net loss accepted by the DM in the face of such a prior.

In an applied context we can interpret this rule as a policy maker who must combine the judgments of a group of advisers with different credibility. They agree about the goals described by the preference over consequences (tastes), but each of them has a prior. The opinion of each member has, potentially, a different weight in the overall decision and the policy maker is able to exhibit a rank between two acts if, and only if, there is a weighted unanimity among the advisers.

Now we introduce the notion of dynamic consistency in a context with partial resolution of uncertainty. Recall that Savage (1954) proposed this idea in axiom P2. However, he models conditional preferences in a static framework. P2 is akin to require that the choice between f and g given E is determined if the acts are equal in case E does not occur and do not matter what they are equal to on E^c. Some models that assume such conditional preferences as primitive and provide definitions like ours are Ghirardato (2002) and Machina and Schmeidler (1992).

Definition 8. *Given an event $E \in \mathcal{R}_{\succsim}$, we say that $\succsim_E$ is the dynamic consistent update of $\succsim$ if for all $f, g \in \mathcal{F}$:*

$$f \succsim_E g \Leftrightarrow fEg \succsim g.$$

[13]It is interesting to note that the ambiguity index η has the same properties of the cost function $c : \Delta \to [0,\infty]$ characterized by MMR for the class of variational preferences.

[14]In fact, a VBP is a Bewley preference if, and only if, it satisfies transitivity, which is equivalent to satisfy independence.

When the pairing $(\succsim, \succsim_E)$ satisfies dynamic consistency the DM weakly prefers act f to g on E and g on the complement if, and only if, he weakly prefers f to g after learning that event E occurs. That is, if a certain course of action was judged *ex ante* to be optimal, the same course of action should be implemented after arrival of information. Next, we present and discuss the first result of our paper.

Theorem 9. *Let the binary relation $\succsim$ be a variational Bewley preference represented by the pair (u, η) and $E \in \mathcal{R}_{\succsim}$. The conditional preference $\succsim_E$ is the dynamic consistent update of $\succsim$ if, and only if, for all $f, g \in \mathcal{F}$,*

$$f \succsim_E g \Leftrightarrow \int_E u(f)dp_E + \eta_E(p_E) \geq \int_E u(g)dp_E \text{ for all } p_E \in \Delta(E),$$

where

$$\eta_E(p_E) = \min_{\{p \in \{\eta < \infty\}: p(\cdot|E) = p_E\}} \frac{\eta(p)}{p(E)}. \tag{1}$$

Therefore, given a relevant event, the updated preference satisfy dynamic consistency if, and only if, it belongs to the same class of preferences of the unconditional, preserves its affine utility over consequences and the ambiguity index follows the full Bayesian update. So, after receiving some information, the DM takes the *ex ante* justification based on a weighted unanimity rule and update their arguments taking into account all the previous ones in the light of the new information. The updating rule given by equation (1) was proposed by Li (2020) for the class of variational preferences of MMR. Li (2015) provides its axiomatic characterization and we will discuss later how our paper relates with that.

In particular, this result also allows us to conclude that the class of VBP satisfy sequential consistency in the sense of Sarin and Wakker (1998). That is, the conditional preference $\succsim_E$ belong to the same class as the unconditional preferences $\succsim$.[15] Now we present an example capturing the previous result.

Example 10. *Assume that $\succsim$ is a VBP that satisfies the following decision rule: for all $f, g \in \mathcal{F}$*

$$f \succsim g \Leftrightarrow \int u(f)dp + \theta D_\phi(p \parallel q) \geq \int u(g)dp \text{ for all } p \in \Delta,\ \theta > 0$$

where

$$D_\phi(p \parallel q) = \begin{cases} \int \phi\left(\frac{dp}{dq}\right) dq, & \text{if } p \in \Delta^\sigma(q), \\ \infty, & \text{otherwise.} \end{cases}$$

[15]Note that the notion of sequential consistency is completely independent of any form of dynamic consistency.

Note that we are concerned about a special case of divergences where the weight is uniform.[16] *Given a relevant event $E \in \mathcal{R}_{\succsim}$, the full Bayesian update of the ambiguity index function is given by $\eta_E(p_E) = \theta D_\phi(p_E \parallel q_E)$. Therefore, by Theorem 9, the conditional relation $\succsim_E$ is represented by a VBP characterized by the pair (u, η_E).*

4 Conditional Variational Preferences and Forced Choices

In this section, we investigate the problem of finding an *ex post* forced choice relation from a VBP, after observing a partial resolution of uncertainty. Formally, we want to characterize and *ex post* relation that is monotone, continuous and a weak order. Recall that a binary relation $\succsim$ on $\mathcal{F}$ is *monotone* if for any $f, g \in \mathcal{F}$, $f(s) \succsim g(s)\ \forall s \in S$ implies $f \succsim g$ and it is a *weak order* if it is *complete* and *transitive.*[17] Moreover, we impose that these relation must preserve the attitude towards uncertainty of the *ex ante* VBP relation, which presupposes some comparative notion.

In order to do that we introduce our notion of comparative ambiguity attitude in a context with an intermediary period where the DM receives some (relevant) information. First, lets start with the static case. As noted by Faro, Ok and Riella (2018), the comparative notion proposed by Ghirardato and Marinacci (2002) could lead to some problems when dealing with incomplete preferences.[18] The idea is that with incomplete preferences the problem of comparative ambiguity attitude gains an additional dimension that the latter authors does not deals with. Hence, we adopt the static definition studied by Faro, Ok and Riella (2018) for incomplete preferences that avoid this concern.

Definition 11. *Given two binary relations over acts $\succsim^1, \succsim^2 \subseteq \mathcal{F} \times \mathcal{F}$, we say that a binary relation $\succsim^1$ is more ambiguity averse than the binary relation $\succsim^2$ if they induce the same preference over constant acts*[19] *and, for every act $f \in \mathcal{F}$ and constant act $x \in X$,*

$$f \succsim^1 x \Rightarrow f \succsim^2 x.$$

[16]Uniform weights guarantees probabilistic sophistication for the class of variational preferences of MMR.

[17]Recall that: $\succsim$ is complete if for all $f, g \in \mathcal{F}$, $f \succsim g$ and/or $g \succsim f$; $\succsim$ is transitive if for all $f, g, h \in \mathcal{F}$, if $f \succsim g$ and $g \succsim h$ then $f \succsim h$. Also, note that a reflexive (axiom 1) and unambiguous transitive (axiom 2) preference satisfies monotonicity.

[18]For example, according to the definition of comparative ambiguity attitude of Ghirardato and Marinacci (2002) it is possible to show that the SEU preference is more ambiguity averse than the Bewley preference. This does not make sense because it seems generally agreed that SEU is used as the benchmark for ambiguity neutrality.

[19]We say that two binary relations induce the same preference over constant acts if for every $x, y \in X$, $x \succsim^1 y \Leftrightarrow x \succsim^2 y$.

Note that, two binary relations that satisfies the comparative ambiguity attitude above and admits an expected utility representation on X can be represented by the same utility index on consequences.[20] Hence, we only compare ambiguity attitude among DM's that have the same risk attitude.[21] Note that definition 11 and the one proposed by Ghirardato and Marinacci (2002) are equivalent when dealing with complete preferences.

Second, lets choose a benchmark to measure absolute ambiguity attitude. Building on Ghirardato and Marinacci (2002), we adopt the class of subjective expected utility (SEU) of Savage (1954) and Anscombe and Aumann (1963) as the benchmark for ambiguity neutrality. Recall that, given an utility v over consequences and a probability measure $q \in \Delta$, we say that $\succsim_{(v,q)}$ on $\mathcal{F}$ is a SEU preference if it is represented by $U(f) = \int v(f)\, dq$.

Definition 12. *We say that a binary relation $\succsim^1$ is ambiguity averse if $\succsim^1$ is more ambiguity averse than some SEU preference $\succsim_{(v,q)}$.*

We may also define ambiguity seeking in the natural way.

Lemma 13. *Let the binary relation $\succsim$ be a VBP represented by the pair (u, η), $\succsim$ is more ambiguity averse than some SEU preference $\succsim_{(v,q)}$. In fact, we can take $v = u$ and $q \in \{\eta = 0\}$.*

Third, we extend the definition of comparative ambiguity attitude to a dynamic context.

Definition 14. *Given $E \in \mathcal{R}_{\succsim^1}$, let $\succsim^1_E$ be a conditional weak order on $\mathcal{F}$ and $\succsim^2$ be an unconditional reflexive binary relation on $\mathcal{F}$.*

(i) We say that $\succsim^1_E$ exhibit more ambiguity aversion *than the ex ante $\succsim^2$, after learning E if, for all $f \in \mathcal{F}$ and $x \in X$,*

$$f \succsim^1_E x \Rightarrow fEx \succsim^2 x.$$

(ii) We say that $\succsim^1_E$ exhibit the same the ambiguity attitude *of the ex ante $\succsim^2$, after learning E if, for all $f \in \mathcal{F}$ and $x \in X$,*

$$f \succsim^1_E x \Leftrightarrow fEx \succsim^2 x.$$

The first part of the definition tell us how the conditional preference restrict the unconditional one. It can be viewed as a generalization of axiom 9 of Pires (2002; p. 143). Moreover, if both $\succsim^1_E$ and $\succsim^2$ belongs to the same class of preferences, then we recover the coherence property introduced by Pires.

[20] This is a consequence of Corollary B.3 of Ghirardato, Maccheroni and Marinacci (2004; p. 163).

[21] Recently, Wang (2020) relax that allowing for separation between ambiguity and risk attitude.

The intuition behind exhibiting more ambiguity aversion is that $\succsim^1_E$ increases the negative attitude towards ambiguity of $\succsim^2$, after learning E. In order to illustrate the meaning of that, we present an example about the relation between a VBP and a variational preference and a special case with a Bewley and a Maxmin expected utility (MEU) preference of Gilboa and Schmeidler (1989). We denote by $\succsim^*$ the unconditional reflexive binary relation $\succsim^2$ and by $\succsim^\#_E$ its conditional forced choice relation after observing E, by $\succsim^1_E$.

Example 15. *Given $E \in \mathcal{R}_{\succsim^*}$, let $\succsim^*$ be a Bewley preference represented by $(u, \mathcal{C})$ and $\succsim_E$ be a MEU preference represented by $(u, \mathcal{C}^{\langle E \rangle})$ where $\mathcal{C}^{\langle E \rangle} \subseteq \Delta(E)$. The conditional relation $\succsim_E$ exhibit more ambiguity aversion than the unconditional relation $\succsim^*$, after learning E if, and only if, $\mathcal{C}^E \subseteq \mathcal{C}^{\langle E \rangle}$.*

More generally, given $E \in \mathcal{R}_{\succsim^*}$, let $\succsim^*$ be a VBP represented by (u, η) and $\succsim_E$ be a variational preference represented by $(u, \eta^{\langle E \rangle})$ where $\eta^{\langle E \rangle} : \Delta(E) \to [0, \infty]$. The conditional relation $\succsim_E$ exhibit more ambiguity aversion than the unconditional relation $\succsim^*$, after learning E if, and only if, $\eta^{\langle E \rangle} \leq \eta_E$.

The second part of definition 14 establishes how the conditional and the unconditional preferences are determined simultaneously once we fix a relevant event. Preserving ambiguity attitude by updating on a relevant event is what we need to fully characterize an unique forced choice relation that is complete, transitive continuous and monotone from an unconditional VBP. The following result formalizes this idea.

Theorem 16. *Let the binary relation $\succsim^*$ be a variational Bewley represented by the pair (u, η) and $\succsim^\#_E$ be a monotone and continuous weak order. The conditional preference $\succsim^\#_E$ exhibit the same the ambiguity attitude of the ex ante $\succsim^*$, after learning $E \in \mathcal{R}_{\succsim^*}$ if, and only if, $\succsim^\#_E$ is a variational preference represented by the pair (u, η_E), that is, for all $f, g \in \mathcal{F}$,*

$$f \succsim^\#_E g \Leftrightarrow \min_{p_E \in \Delta(E)} \left[\int u(f) dp_E + \eta_E(p_E) \right] \geq \min_{p_E \in \Delta(E)} \left[\int u(g) dp_E + \eta_E(p_E) \right]$$

where $\eta_E(p_E)$ is the full Bayesian update of $\eta(p)$ given by (1).

Theorem 16 provides a full characterization of the conditional relation $\succsim^\#_E$ as the unique *ex post* forced choice relation that satisfies completeness, transitivity,[22] monotonicity and continuity from a VBP $\succsim^*$ represented by the pair (u, η). The *ex post* relation has a variational preference representation[23] with the pair (u, η_E) where u is the

[22]Note that $\succsim^\#$ could revert some ranking to avoid lack of transitivity of $\succsim^*$.

[23]It worth noting that for specific functionals of η for the variational representation of MMR it is possible to recover three important models in the literature: MEU preference of Gilboa and Schmeidler (1989), mean-variance preferences of Markowitz (1952) and Tobin (1958) (on the domain of monotonicity) and multiplier preferences of Hansen and Sargent (2000, 2001).

same affine utility over consequences than $\succsim^*$, and η_E (the *ex post* ambiguity index) is derived via a generalization of the full Bayesian update of η. For instance, a special case is when the first relation is a Bewley preference represented by $(u, \mathcal{C})$ and the second a MEU preference represented by $(u, \mathcal{C}^E)$. Also, it is worth noting that both class of preferences considered here are ambiguity averse.[24] Hence, we could be more precise and say that $\succsim_E^\#$ exhibit the same the ambiguity aversion of the *ex ante* $\succsim^*$, after learning E. Note that we assume dynamic consistency as a compelling requirement only for ranking of acts captured by the VBP and not for forced choices.[25]

Finally, we could interpret the result as representing a DM that has incomplete preferences represented by a VBP that will consider the criterion given by variational preferences if he is forced to make a choice after learning that the true state of nature belongs to a relevant event.

5 Related Literature

In this section we clarify the connections between Theorems 9 and 16 and the works of Faro (2015), Li (2015), Faro and Lefort (2019) and Gilboa *et al.* (2010).

Note that if we combine Theorems 9 and 16 we have that $\succsim_E^*$ is a VBP and $\succsim_E^\#$ a variational preference, both represented by the pair (u, η_E) where (u, η) are the utility and ambiguity index from the unconditional relation and η_E is defined by equation (1). Theorem 14 of Faro (2015; p. 714) formalizes that this is equivalent to $\succsim_E^*$ and $\succsim_E^\#$ jointly satisfy both weak consistency[26] and default to certainty.[27] The former axiom was proposed by Faro and the latter by Gilboa *et al.* (2010). Therefore, our notion of exhibiting the same ambiguity attitude when $E = S$ is exactly the combination of the previous axioms.

Alternatively, Theorem 4 of Li (2015; p. 28) obtains the updating rule of (1) assuming that the unconditional preference admits a variational representation and satisfies two axioms: stable risk preferences[28] and conditional certainty equivalent consistency.[29] From

[24]See Proposition 7 of MMR (p. 1457).

[25]In the presence of ambiguity it is not clear that dynamic consistency must hold. For instance, for the class of MEU preferences, Epstein and Schneider (2003) showed that this is not true unless the set of priors is rectangular. Ghirardato, Maccheroni and Marinacci (2008) claim that dynamic consistency is a compelling property only for comparisons of acts that are not affected by the possible presence of ambiguity.

[26]A pair of binary relations $(\succsim^*, \succsim^\#)$ satisfies consistency when for any $f, g \in \mathcal{F}$, $f \succsim^* g$ implies $f \succsim^\# g$, weak consistency just imposing that g should be a constant act.

[27]A pair of binary relations $(\succsim^*, \succsim^\#)$ satisfies default to certainty when for any $f, g \in \mathcal{F}$, if not $f \succsim^* x$, then $x \succ^\# f$.

[28]Stable risk preferences means the DM's preferences over constant acts are not affected by anticipated information about events.

[29]Conditional certainty equivalent consistency means that dynamic consistency holds when considering

other perspective, Theorem 1 of Faro and Lefort (2019; p.7) is a particular case of our Theorem 16 when the unconditional relation is a Bewley preference (i.e., a VBP with $\eta = \delta_{\mathcal{C}}$) and its completion a MEU (a particular case of the variational preferences when η is the indicator function). Hence, the pair $(\succsim^*, \succsim_E^\#)$ jointly satisfies weak intertemporal consistency[30] and intertemporal default to certainty.[31] The former, a weaker form of the intertemporal consistency axiom proposed by Faro and Lefort (2019) and the latter also proposed by the same authors.

6 Conclusion

In this paper we propose a dynamic consistent updating rule for the class of variational Bewley preferences. Theorem 9 shows that the conditional relation still in the same class of preferences, the utility over consequences is preserved from the unconditional one and the ambiguity index follows a generalized form of full Bayesian update. For our second result, we address the problem of modeling forced choices after a partial resolution of uncertainty.

We consider a DM characterized by two binary relations, where the first preference, denoted by $\succsim^*$, is an unconditional VBP and the second preference, denoted by $\succsim_E^\#$, is a conditional preference w.r.t. a relevant event $E \in \mathcal{R}_{\succsim^*}$, given by a monotone and continuous weak order. After introducing the notion of comparative ambiguity attitude, Theorem 16 shows that $\succsim_E^\#$ exhibit the same the ambiguity attitude of the *ex ante* $\succsim^*$, after learning E if, and only if, it belongs to the class of variational preferences, the utility over consequences is preserved from $\succsim^*$ and the ambiguity index follows the same updating rule that characterizes VBP. Thus, our result can be viewed as a novel foundation for the updating rule of variational preferences proposed by Li (2020).

Appendix A

Proof of the Results in the Main Text

Remark 17. *We note that, since we assume that E is $\succsim$-relevant then E is $\succsim$-non-null, by Lemma 2 of Li (2015; p. 39) it follows that $p(E) > 0$, $\forall p \in \eta^{-1}(0) = \{p \in \Delta(S) :$*

indifference among an uncertain result on E and a certain consequence on the complement. This axiom was first introduced by Pires (2002) for the class of MEU and later used by Eichberger *et al.* (2007) for the Choquet expected utility model of Schmeidler (1989).

[30] A pair of binary relations $(\succsim^*, \succsim_E^\#)$ satisfies intertemporal consistency when for any $f, g \in \mathcal{F}$, $fEg \succsim^* g$ implies $f \succsim_E^\# g$, weak intertemporal consistency just imposing that g should be a constant act.

[31] This axiom establishes how to link an *ex ante* unconditional preference and an *ex post* conditional preference taking into account a special class of acts.

$\eta(p) = 0\}$. *Finally, by Lemma 6 of Li (2015; p. 49), we obtain*

$$\eta_E(p_E) = \min\left\{\frac{\eta(p)}{p(E)} : p \in \{\eta < \infty\} \textit{ and } p(\cdot|E) = p_E\right\}.$$

Proof of Theorem 9. The conditional preference $\succsim_E$ is the dynamic consistent update of $\succsim$ and by definition 8 we have that for any $f, g \in \mathcal{F}$ and given $E \in \mathcal{R}_{\succsim}$, $f \succsim_E g \Leftrightarrow fEg \succsim g$. Since $\succsim$ is a VBP represented by the pair (u, η) we have: for all $f, g \in \mathcal{F}$,

$$\begin{aligned} f \succsim_E g &\Leftrightarrow \int u(fEg)dp + \eta(p) \geq \int u(g)dp, \forall p \in \{\eta < \infty\} \\ &\Leftrightarrow \int_E u(f)dp + \eta(p) \geq \int_E u(g)dp, \forall p \in \{\eta < \infty\} \\ &\Leftrightarrow \int_E u(f)dp_E + \frac{\eta(p)}{p(E)} \geq \int_E u(g)dp_E, \forall p \in \{\eta < \infty\} \text{ s.t. } p(E) > 0 \\ &\Leftrightarrow \min_{\{p\in\{\eta<\infty\}:p(\cdot|E)=p_E\}} \frac{\eta(p)}{p(E)} \geq \min_{\{p\in\{\eta<\infty\}:p(\cdot|E)=p_E\}} \left[\int_E u(g)dp_E - \int_E u(f)dp_E\right], \\ &\qquad\qquad \forall p \in \{\eta < \infty\} \text{ s.t. } p(E) > 0 \\ &\Leftrightarrow \eta_E(p_E) \geq \int_E u(g)dp_E - \int_E u(f)dp_E, \forall p \in \{\eta < \infty\} \text{ s.t. } p(E) > 0 \\ &\Leftrightarrow \int_E u(f)dp_E + \eta_E(p_E) \geq \int_E u(g)dp_E, \forall p_E \in \{\eta_E < \infty\}, \end{aligned}$$

which concludes de proof. $\square$

Proof of Example 10. Applying the updating rule (1),

$$\begin{aligned} \eta_E(p_E) &= \min_{\{p\in\{\eta<\infty\}:p(\cdot|E)=p_E\}} \frac{\theta}{p(E)} \int_S \phi\left(\frac{dp}{dq}\right) dq \\ &= \min_{\{p\in\{\eta<\infty\}:p(\cdot|E)=p_E\}} \frac{\theta}{p(E)} \left\{\left[\int_E \phi\left(\frac{dp}{dq}\right) dq_E\right] q(E) + \left[\int_{E^c} \phi\left(\frac{dp}{dq}\right) dq_{E^c}\right] q(E^c)\right\} \\ &= \min_{\{p\in\{\eta<\infty\}:p(\cdot|E)=p_E\}} \frac{\theta}{p(E)} \left\{\left[\int_E \phi\left(\frac{dp_E}{dq_E}\frac{p(E)}{q(E)}\right) dq_E\right] q(E) + \right. \\ &\qquad\qquad \left. + \left[\int_{E^c} \phi\left(\frac{dp_{E^c}}{dq_{E^c}}\frac{p(E^c)}{q(E^c)}\right) dq_{E^c}\right] q(E^c)\right\} \\ &= \theta \int_E \phi\left(\frac{dp_E}{dq_E}\right) dq_E \end{aligned}$$

In the last step, we choose a minimizing p that satisfies $p(E) = q(E)$ and $p(\cdot|E^c) = q(\cdot|E^c)$. So $\eta_E(p_E) = \theta D_\phi(p_E \parallel q_E)$. $\square$

Proof of Lemma 13. Theorem 1 of Faro (2015; p. 706) showed that a VBP represented by the pair (u, η) admits a representation *a la* von Neumann-Morgenstern when restricted to constant acts. By assumption, $\succsim$ is more ambiguity averse than a SEU preference $\succsim_{(v,q)}$. Hence, Corollary B.3 of Ghirardato, Maccheroni and Marinacci (2004;

p. 163) implies that we can take $u = v$. Also, since $\{\eta = 0\} \neq \emptyset$ lets choose $q \in \{\eta = 0\}$. Given $f \in \mathcal{F}$ and $x \in X$,

$$f \succsim x \Rightarrow \int u(f)dp + \eta(p) \geq u(x),\ \forall p \in \{\eta < \infty\} \Rightarrow \int u(f)dq \geq u(x)$$

Hence, $f \succsim_{(u,q)} x$. $\square$

Proof of Example 15.

($\Leftarrow$) Since $\succsim_E^{\#}$ is a variational preference represented by $(u, \eta^{\langle E \rangle})$ we have: for all $f \in \mathcal{F}$ and $x \in X$,

$$f \succsim_E^{\#} x \Leftrightarrow \min_{p_E \in \{\eta^{\langle E \rangle} < \infty\}} \left[\int_E u(f)dp_E + \eta^{\langle E \rangle}(p_E) \right] \geq u(x)$$

since $\eta^{\langle E \rangle}$ is grounded. Therefore,

$$\min_{p_E \in \{\eta^{\langle E \rangle} < \infty\}} \left[\int_E u(f)dp_E + \eta^{\langle E \rangle}(p_E) \right] \geq u(x) \Leftrightarrow$$
$$\int_E u(f)dp_E + \eta^{\langle E \rangle}(p_E) \geq u(x),\ \forall p_E \in \{\eta^{\langle E \rangle} < \infty\}$$

by assumption

$$\eta^{\langle E \rangle} \leq \eta_E \Rightarrow \{\eta_E < \infty\} \subseteq \{\eta^{\langle E \rangle} < \infty\},$$

which implies,

$$\int_E u(f)dp_E + \eta^{\langle E \rangle}(p_E) \geq u(x),\ \forall p_E \in \{\eta^{\langle E \rangle} < \infty\} \Rightarrow$$
$$\int_E u(f)dp_E + \eta_E(p_E) \geq u(x),\ \forall p_E \in \{\eta_E < \infty\}.$$

Therefore,

$$\int_E u(f)dp_E + \min_{\{p \in \{\eta < \infty\} : p(\cdot|E) = p_E\}} \frac{\eta(p)}{p(E)} \geq u(x),\ \forall p_E \in \{\eta_E < \infty\},$$

which implies,

$$\frac{1}{p(E)} \int_E u(f)dp + \frac{\eta(p)}{p(E)} \geq u(x),\ \forall p \in \{\eta < \infty\} \Rightarrow$$
$$\int u(fEx)dp + \eta(p) \geq u(x),\ \forall p \in \{\eta < \infty\}.$$

Hence, $fEx \succsim^* x$.

($\Rightarrow$) Assume that $\succsim_E$ exhibit more ambiguity aversion than the unconditional relation $\succsim^*$, after learning E. By definition 11, for all $f \in \mathcal{F}$ and $x \in X$,

$$f \succsim_E x \Rightarrow fEx \succsim^* x. \tag{2}$$

Now consider the conditional preference relation $\succsim_E^{\#}$ that exhibit the same the ambiguity attitude of $\succsim^*$, after learning E. By definition 11, for all $f \in \mathcal{F}$ and $x \in X$,

$$f \succsim_E^{\#} x \Leftrightarrow fEx \succsim^* x. \tag{3}$$

Equations (2) and (3) implies that for all $f \in \mathcal{F}$ and $x \in X$,

$$f \succsim_E x \Rightarrow f \succsim_E^{\#} x. \tag{4}$$

Our Theorem 16 implies that $\succsim_E^{\#}$ is a variational preference represented by (u, η_E), where η_E is given by equation (1). By assumption $\succsim_E$ is a variational preference represented by $(u, \eta^{\langle E \rangle})$. Hence, equation (4) implies, by Proposition 8 of MMR (p. 1457), that $\eta^{\langle E \rangle} \leq \eta_E$. $\square$

Proof of Theorem 16.

($\Leftarrow$) Given $f \in \mathcal{F}$ and $x \in X$, assume that $fEx \succsim^* x$. Thus,

$$\int_S u(fEx)dp + \eta(p) \geq u(x),\ \forall p \in \{\eta < \infty\} \Rightarrow$$
$$\int_E u(f)dp + \eta(p) \geq p(E)u(x),\ \forall p \in \{\eta < \infty\}.$$

If by contradiction, we have not $f \succsim_E^{\#} x$ then, since $\succsim_E^{\#}$ is complete, we have $x \succ_E^{\#} f$. Since $\succsim_E^{\#}$ is a variational preference represented by (u, η_E) it follows that

$$u(x) > \min_{p_E \in \{\eta_E < \infty\}} \left\{ \int_E u(f)dp_E + \eta_E(p_E) \right\} \Rightarrow u(x) > \int_E u(f)dp_E^* + \eta_E(p_E^*)$$

where,

$$p_E^* \in \arg \min_{p_E \in \{\eta_E < \infty\}} \left\{ \int_E u(f)dp_E + \eta_E(p_E) \right\}.$$

Hence, combining the inequalities above: $\forall \bar{p} \in \{\eta < \infty\}$ s.t. $\bar{p}(\cdot|E) = p_E^*$

$$\int_E u(f)d\bar{p} + \eta(\bar{p}) \geq \bar{p}(E)u(x) > \bar{p}(E) \int_E u(f)dp_E^* + \bar{p}(E)\eta_E(p_E^*).$$

Therefore,

$$\frac{\eta(\bar{p})}{\bar{p}(E)} > \eta_E(p_E^*),\ \forall \bar{p} \in \{\eta < \infty\} \text{ s.t. } \bar{p}(\cdot|E) = p_E^*,$$

which is a contradiction because

$$\eta_E(p_E) = \min_{\{p \in \{\eta < \infty\} : p(\cdot|E) = p_E\}} \frac{\eta(p)}{p(E)}.$$

Now, assume that $f \succsim_E^\# x$ (i.e., not $x \succ_E^\# f$). Hence,

$$\min_{p_E \in \{\eta_E < \infty\}} \left\{ \int_E u(f) dp_E + \eta_E(p_E) \right\} \geq u(x).$$

So, $\forall p_E \in \{\eta_E < \infty\}$,

$$\int_E u(f) dp_E + \eta_E(p_E) \geq u(x).$$

Now, by contradiction, suppose that not $fEx \succsim^* x$. Hence, $\exists p^* \in \{\eta < \infty\}$ s.t.

$$\int u(fEx) dp^* + \eta(p^*) < u(x) \Rightarrow \int_E u(f) dp^* + \eta(p^*) < p^*(E) u(x).$$

If $p^*(E) = 0$ then $\eta(p^*) < 0$, which is impossible, so $p^*(E) > 0$. Moreover,

$$u(x) > \frac{1}{p^*(E)} \int_E u(f) dp^* + \frac{\eta(p^*)}{p^*(E)} = \int_E u(f) dp_E^* + \frac{\eta(p^*)}{p^*(E)} \geq \int_E u(f) dp_E^* + \eta_E(p_E^*),$$

a contradiction.

($\Rightarrow$) Let $\succsim^*$ be a VBP represented by (u, η) and $\succsim_E^\#$ be a weak order, continuous and monotone. Consider $E \in \mathcal{R}_{\succsim^*}$ and assume that the conditional preference $\succsim_E^\#$ exhibit the same the ambiguity attitude of the *ex ante* $\succsim^*$, after learning $E \in \mathcal{R}_{\succsim^*}$, i.e., for all $f \in \mathcal{F}$ and $x \in X$,

$$f \succsim_E^\# x \Leftrightarrow fEx \succsim^* x.$$

First, $\succsim^*$ and $\succsim_E^\#$ are the same relations over the subset of constant acts:

$$x \succsim^* y \Rightarrow xEy \succsim^* y \Rightarrow x \succsim_E^\# y$$
$$\text{not } y \succsim^* x \Rightarrow x \succ^* y \Rightarrow x \succ_E^\# y \Rightarrow \text{ not } y \succsim_E^\# x.$$

Therefore, $\succsim^*$ and $\succsim_E^\#$ coincide on X and the mapping $x \mapsto u(x)$ also represents $\succsim_E^\#$ on X. Since $\succsim_E^\#$ satisfies monotonicity and continuity, for any act $f \in \mathcal{F}$ we can find $x_f \in X$ that is the certainty equivalent of f with respect to $\succsim_E^\#$. By our assumption, if not $fEx_f \succsim^* x_f$ then $x_f \succ_E^\# f$, a contradiction. Thus, $fEx_f \succsim^* x_f$ and since $\succsim^*$ has a VBP representation (u, η), it follows that for any $p \in \{\eta < \infty\}$,

$$\int_S u(fEx_f) dp + \eta(p) \geq \int_S u(x_f) dp \Rightarrow \int_E u(f) dp + \eta(p) \geq u(x_f) p(E)$$
$$\Rightarrow \int_E u(f) dp_E + \frac{\eta(p)}{p(E)} \geq u(x_f)$$

Moreover,

$$\int_E u(f) dp_E + \min_{\{p \in \{\eta < \infty\} : p(\cdot|E) = p_E\}} \frac{\eta(p)}{p(E)} \geq u(x_f)$$
$$\Rightarrow \int_E u(f) dp_E + \eta_E(p_E) \geq u(x_f), \ \forall p_E \in \{\eta_E < \infty\}$$
$$\Rightarrow \min_{p_E \in \{\eta_E < \infty\}} \left[\int_E u(f) dp_E + \eta_E(p_E) \right] \geq u(x_f)$$

If the strict inequality holds, then there exists $y \in X$ such that

$$u(x_f) < u(y) < \min_{p_E \in \{\eta_E < \infty\}} \left[\int_E u(f) dp_E + \eta_E(p_E) \right]$$

Since $\succsim^*$ and $\succsim_E^\#$ coincide on X and $\succsim_E^\#$ is a weak order, we have that $u(x_f) < u(y) \Rightarrow y \succ_E^\# x_f$. We also have that

$$\begin{aligned} & u(y) < \int_E u(f) dp_E + \eta_E(p_E),\ \forall p_E \in \{\eta_E < \infty\} \\ \Leftrightarrow\ & u(y) < \int_E u(f) dp_E + \frac{\eta(p)}{p(E)},\ \forall p \in \{\eta < \infty\} \\ \Leftrightarrow\ & u(y) < \int_S u(fEy) dp + \eta(p),\ \forall p \in \{\eta < \infty\}. \end{aligned}$$

That is, $fEy \succsim^* y$ and by our assumption $f \succsim_E^\# y$. Since $\succsim_E^\#$ is transitive, $f \succsim_E^\# y$ and $y \succ_E^\# x_f$ implies that $f \succ_E^\# x_f$, which is impossible. Hence,

$$u(x_f) = \min_{p_E \in \{\eta_E < \infty\}} \left\{ \int_E u(f) dp_E + \eta_E(p_E) \right\},$$

i.e., $\succsim_E^\#$ is a variational preference represented by (u, η_E).

In order to conclude, since $\succsim^*$ and $\succsim_E^\#$ are the same on constant acts and $\succsim^*$ satisfies Unboundedness, by Proposition 6 of MMR (p. 1457), η_E is unique. □

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