Independence and Variational Bewley Preferences: A Note

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Abstract

This note studies some alternatives and weak versions of the Independence axiom in a decision theoretic framework under uncertainty. We propose a characterization of this axiom using a property called Weight Independence. Moreover we study how the Independence axiom is related with the Variational Bewley model of Faro (2015). We show that Variational Bewley preferences satisfy a weaker form of independence called Independence for Constant Weights. This topic gives us the opportunity to discuss the pioneeristic contributions of David Schmeidler on the weakening of the Independence axiom.

KEYWORDS: Ambiguity, Knightian uncertainty, independence, incomplete preferences, Variational preferences.

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1 Introduction

The Independence axiom is one of the most important condition in the theory of decision under risk and under uncertainty. Consider a decision maker (DM) with a preference relation over functions from states of the world to lotteries.[1] As usual, we denote this preference relation with the symbol $\succsim$ and we call these functions *acts*. When we write $f \succsim g$ we mean "act f is at least as good as act g". Then, the Independence axiom says the following

Independence. For all acts f, g, and h and $\alpha \in (0,1)$ $f \succsim g \Leftrightarrow \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$.

In words, this condition says that if an act f is preferred to another act g, then the lottery that with probability α delivers f and with probability $1-\alpha$ delivers h should be better than the one that with probability α delivers g and with probability $1-\alpha$ delivers h. The reasoning can be done the other way around too, as the condition is stated with a double implication.

This axiom has a very appealing interpretation: loosely speaking it says that the DM's preferences should be based only on aspects where acts differ. To elaborate more, suppose that the DM prefers f to g. Then the DM should prefer to get f with probability α rather than g with probability α, if, with probability $(1-\alpha)$ she ends up receiving the same act, namely h. Independence says that this reasoning should be valid for any α and any h. On the other hand, suppose that for some acts f, g, h and some constant $\alpha \in (0,1)$ the preferences of the DM are such that $\alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$. Since with probability $(1-\alpha)$ the DM gets the same act in both lotteries, namely h, the "direction" of preferences should be based only upon acts f and g, and hence $f \succsim g$.

This explanation seems straightforward, but actually it tacitly implies dynamic consistency and reduction of compounded lotteries (see Karni and Schmeidler (1991) and Gilboa (2009)).

[1] These are Anscombe and Aumann (1963) acts. The formal framework will be introduced in Section 2.

Moreover the descriptive validity of the Independence axiom has been challenged by several known behavioral experiments. For instance, Fishburn and Wakker (1995) write: "At first sight, the condition seems compelling as a normative principle. Nevertheless, the famous Allais and Ellsberg paradoxes have called it into question. Independence has been staunchly defended by some and severely criticized and modified by others, but it is seldom ignored.".

The Independence axiom is a fundamental property of preferences in order to get the Expected Utility (EU) model. For instance, Hara *et al.* (2019) show that a reflexive binary relation (over risky prospects) that satisfies the Independence axiom, but that it is not necessarily transitive, complete or continuous, admits a form of EU representation. It is clear that, if one is willing to depart from EU (in order to explain, say, the Ellsberg (1961) paradox), the Independence axiom should be reconsidered.

David Schmeidler pioneered the literature that studies possible ways to weaken the Independence axiom. This is essential in order to obtain a more general theory that can accommodate some behavioral features inconsistent with EU, and especially Ellsberg (1961) paradox. In his seminal paper, Schmeidler (1989), he imposed independence only to acts that are pairwise comonotonic. Two acts are comonotonic if they "vary in the same direction". More formally, f and g are comonotonic if there are no states of the world s and s' such that $f(s) \succ f(s')$ and $g(s') \succ g(s)$. If acts are comonotonic, no hedging considerations can be done by the DM when mixing two acts, i.e. mixing comonotonic acts does not offer any protection against uncertainty. As Schmeidler (1989) puts it: "If f, g, and h are pairwise comonotonic, then the comparison of f to g is not very different from the comparison of $[\alpha f + (1-\alpha)h]$ to $[\alpha g + (1-\alpha)h]$. Hence the decision maker can accept the validity of the implication: $[f \succsim g \Leftrightarrow \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h]$, without fear of running into a contradiction". Schmeidler (1989) calls independence over comonotonic acts Comonotonic Independence.

The second key contribution of David Schmeidler about Independence lies on the definition of the axiom Certainty-Independence in Gilboa and Schmeidler (1989). This is a further weakening of Comonotonic Independence since it requires independence to hold only when mixing with constant acts. Gilboa and Schmeidler (1989) write: "This axiom seems heuristically more appealing [than the Independence axiom]: a decision maker who prefers f to g can more easily visualize thee mixtures of f and g with a constant h than with an arbitrary one, hence he is less likely to reverse his preferences. An intuitive objection to the standard [I]ndependece axiom is that it ignores the phenomenon of hedging. Like [C]omonotonic [I]ndependence [...], C[ertainty]-independece does not exclude hedging.". The axiom of Certainty-Independence is a key axiom (together with Uncertainty Aversion) in Gilboa and Schmeidler (1989) and it delivers the famous MaxMin Expected Utility model.

This paper wishes to contribute to the literature that studies extensions of the Independence axiom. We do so in a model developed by Faro (2015). This model characterizes preference relations that may fail not only independence, but also completeness and transitivity. In a framework *à la* Anscombe and Aumann (1963), Faro (2015) axiomatized preference relations $\succsim$, called *variational Bewley preferences*, represented by an affine utility function u over consequences and an ambiguity index η, such that

$$f \succsim g \Leftrightarrow \int u(f)\, dp + \eta(p) \geq \int u(g)\, dp \text{ for all prior } p.$$

The representation can be interpreted as a weighted unanimity rule, where the function η reflects the weight given to a prior and higher values of η correspond to priors with less weight.

The axiomatic foundation of variational Bewley preferences is shown by Faro(2015) to rest on a simple set of axioms generalizing the model proposed by Ghirardato *et al.* (2004) for Bewley (2002) incomplete preferences. Bewley's and Faro's model proved to be important tools in order to explain experimental data about choice under uncertainty. In fact, Cettolin and Riedl (2019) provide empirical evidence in which individuals exhibit a choice pattern inconsistent with Completeness and Certainty-Independence. They found that about half of the participant of one of their proposed experiments exhibit behavior coherent with incomplete preferences *à la* Bewley (2002) or Faro (2015).

Since Variational Bewley preferences do not satisfy the Independence axiom in general, we are interested to understand which kind of independence properties they satisfy. In order to do so, we define a condition, called Weight Independence that shows how the Independence axiom implies two forms of independence: with respect to the act h in the mixture and with respect to the mixing constant α.[2] Weight Independence suggests a possible weakening that we call Independence for Constant Weights. This condition relaxes the Independence axiom by requiring independence only with respect to acts in the mixture, but not with respect to the mixing constant. Our main result shows that Variational Bewley preferences may not satisfy Weight Independence, but they do satisfy Independence for Constant Weights.

The rest of the paper is organized as follows. Section 2 introduce the framework, the notation and the relevant mathematics. Section 3 describes Faro (2015) model and axioms. Section 4 present our results about independence. Section 5 concludes.

2 Framework

Consider a set S of *states of nature (world)*, endowed with a σ-algebra Σ of subsets called *events*, and a non-empty set X of *consequences*. We denote by $\mathcal{F}$ the set of all the (simple) *acts*: finite-valued functions $f : S \to X$ which are Σ-measurable[3]. Moreover, we denote by $B_0(\Sigma)$ the set of all simple real-valued Σ-measurable functions $a : S \to \mathbb{R}$. The norm in $B_0(\Sigma)$ is given by $\|a\|_\infty = \sup_{s\in S} |a(s)|$ (called *sup norm*) and will denote by $B(\Sigma)$ the supnorm closure of $B_0(\Sigma)$. In another way, $B_0(\Sigma)$ is the vector space generated by the indicator functions of the elements of Σ, endowed with the supnorm[4]. We denote by $ba(\Sigma)$ the Banach space of all finitely additive set functions on Σ endowed with the total variation norm. This space is isometrically isomorphic to the norm dual of $B_0(\Sigma)$. Note also that the weak* topology $\sigma(ba, B_0)$ of $ba(\Sigma)$ coincides with the event-wise convergence topology. Throughout the paper, we assume that any subset of $ba(\Sigma)$ is endowed with the topology inherited from the weak* topology.

Given a mapping $u : X \to \mathbb{R}$, the function $u(f) : S \to \mathbb{R}$ is defined by $u(f)(s) = u(f(s))$, for all $s \in S$. We note that $u(f) \in B_0(\Sigma)$ whenever f belongs to $\mathcal{F}$.

Let x belong to X, define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. Hence, we can identify X with the set $\mathcal{F}_c$ of the constant acts in $\mathcal{F}$.

Additionally, we assume that the set of consequences X is a convex subset of a vector space. For instance, this is the case if X is the set of all simple lotteries on a set of *outcomes* Z. In fact, it is the classic setting of Anscombe and Aumann (1963) as re-started by Fishburn (1970).

Using the linear structure of X we can define as usual for every $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$ the act:

$$\alpha f + (1-\alpha)g : S \to X$$
$$(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s).$$

Also, given two acts $f, g \in \mathcal{F}$ and an event $E \in \Sigma$ we denote by fEg the act delivering the consequences $f(s)$ in E and $g(s)$ in E^c.

[2] This was noted also by Maccheroni *et al.* (2006) for Certainty-Independence

[3] Let $\succsim_0$ be a binary relation on X, we say that a function $f : S \to X$ is Σ-measurable if, for all $x \in X$, the sets $\{s \in S : f(s) \succsim_0 x\}$ and $\{s \in S : f(s) \succ_0 x\}$ belong to Σ.

[4] We refer to Dunford and Schwartz (1958), section 5 of chapter IV for all details concerning this paragraph.

We denote by $\Delta := \Delta(\Sigma)$ the set of all (finitely additive) probability measures $p : \Sigma \to [0,1]$. We say that a mapping $\eta : \Delta \to [0,\infty]$ is grounded if $\{\eta = 0\} := \{p \in \Delta : \eta(p) = 0\} \neq \emptyset$. The effective domain of $\eta : \Delta \to [0,\infty]$ is defined by $dom(\eta) := \{\eta < \infty\}$. Also, η is lower semicontinuous if $\{\eta \leq r\}$ is closed for each $r \geq 0$.

Functions of the form $\eta : \Delta \to [0,\infty]$ will play a key role in the paper because it will capture the subjective degree of plausibility of each prior for a DM. We denote by $\mathcal{N}(\Delta)$ the class of these functions such that η is grounded, convex and lower semicontinuous. As discussed in the Introduction, a mapping η is called an ambiguity index, as introduced in Maccheroni *et al.* (2006) (MMR from now on).

3 Axioms and Variational Bewley Preferences

The DM's preferences are given by a binary relation $\succsim$ on $\mathcal{F}$. As usual its symmetric and asymmetric components are denoted by $\sim$ and $\succ$. Next, we describe the axioms on $\succsim$ that captures the class of variational Bewley preferences.

A.1–Reflexivity. For all $f \in \mathcal{F}$, $f \succsim f$.

A.2–Unambiguous Transitivity. Suppose $f \succsim g$. If $h(s) \succsim f(s)$ for all s, then $h \succsim g$. Also, if $g(s) \succsim h(s)$ for all s, then $f \succsim h$.

A.3–C-Completeness. For all constant acts x and y, $x \succsim y$ or $y \succsim x$.

A.4–Mixture Continuity. For all $f, g, h \in \mathcal{F}$ the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succsim h\}$ and $\{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha)g\}$ are closed in $[0,1]$.

A.5–Dominance Independence. For all $f_1, f_2, h_1, h_2 \in \mathcal{F}$, and all $\alpha \in (0,1)$,

$$\text{if } f_1 \succsim f_2 \text{ and } h_1 \succsim h_2 \text{ then } \alpha f_1 + (1-\alpha)h_1 \succsim \alpha f_2 + (1-\alpha)h_2.$$

A.6–Unboundedness. There are $x, y \in X$ such that, for each $\alpha \in (0,1)$, there exist $z, \widehat{z} \in X$ such that $\alpha z + (1-\alpha)y \succ x \succ y \succ \alpha\widehat{z} + (1-\alpha)x$.

While A.1 and A.4 are standard axioms, the others are not. These axioms are thoroughly discussed in Faro (2015). We comment them briefly here.

Axiom A.2 relaxes the usual transitivity condition, i.e. for all $f, g, h \in \mathcal{F}$ $f \succsim g$ and $g \succsim h$ imply $f \succsim h$. Here transitivity must hold only when acts are related by state by state dominance. Clearly, a preference relation that satisfies only Unambiguous Transitivity may fail to be transitive in the usual sense.

Axiom A.3 is a weak version of the completeness axiom, i.e. for all $f, g \in \mathcal{F}$ $f \succsim g$ or $g \succsim f$. The full-strength completeness axiom requires the DM to be able to compare all acts in $\mathcal{F}$. By contrast, C-Completeness only says that the DM is able to compare constant acts. Since these acts do not involve any ambiguity, C-Completeness implies that the DM's preferences would be incomplete solely for effect of ambiguity.

Axiom A.6 is a technical axiom. It implies that the image of the utility function $u : X \to \mathbb{R}$ is $u(X) = \mathbb{R}$.

Finally, axiom A.5 imposes some sort of independence on preferences. Let us say that f_1 and h_1 are acts that the DM likes and f_2 and h_2 are acts that the DM dislikes. Then Dominance Independence says, roughly speaking, that a mixture of the two "good acts", should be preferred to the mixture of the two "bad acts" *if the mixing weight is kept constant.* As noticed in Faro (2015), Dominance Independence does not implies the Independence axiom but only the following weaker form

Weak-Independence. For all acts f, g, and h and $\alpha \in (0,1)$ $f \succsim g \Rightarrow \alpha f+(1-\alpha)h \succsim \alpha g+(1-\alpha)h$.

Next, we present the main result of Faro (2015), in which he characterized the representation that results from Axioms A.1–A.6.

Theorem 1 *(Faro, 2015). Let $\succsim$ be a preference relation on the set of acts $\mathcal{F}$. Then the following conditions are equivalent:*

1. *$\succsim$ satisfies assumptions A.1–A.6.*
2. *There exists an affine utility index $u : X \to \mathbb{R}$, with $u(X) = \mathbb{R}$, and a function $\eta : \Delta \to [0,\infty]$ that belongs to $\mathcal{N}(\Delta)$ such that, for all f and g in $\mathcal{F}$,*

$$f \succsim g \Leftrightarrow \int u(f)dp + \eta(p) \geq \int u(g)dp, \forall p \in \Delta.$$

For each u there is a unique $\eta^* : \Delta \to [0,\infty]$ consistent with our representation given by

$$\eta^*(p) = \sup_{(f,g)\in\succsim} \left(\int (u(g) - u(f))\, dp \right), \ \forall p \in \Delta.$$

Theorem 1 generalizes Bewley (2002) model of decision making under uncertainty with incomplete preferences. In Bewley's model a DM ranks acts according to the following unanimity rule

$$f \succsim g \Leftrightarrow \int u(f)dp \geq \int u(g)dp, \forall p \in C \tag{1}$$

where C is a set of probabilities. The model in (1) is recovered when either Transitivity or Independence are added to the axioms that characterize variational Bewley preferences. In this case $C = \{p : \eta^*(p) = 0\}$, see Theorem 5 in Faro (2015).

One interesting aspect of Faro's model is that the ambiguity index η has the same properties of the cost function $c : \Delta \to [0,\infty]$ characterized in MMR's *variational preferences.* We recall that in that model a DM has a (complete and transitive) preference relation over the set of acts $\mathcal{F}$ represented by the functional

$$V(f) = \min_{p\in\Delta} \left\{ \int u(f)dp + c(p) \right\} \tag{2}$$

Despite the behavioral differences between the set of axioms that characterizes variational preferences and variational Bewley preferences, the models share a similar functional representation. Notice in fact that a variational Bewley preference can be described by $f \succsim g \Leftrightarrow \min_{p\in\Delta} \left\{ \int u(f) - u(g)dp + \eta^*(p) \right\} \geq 0$. Actually, the similarity of the representation in Theorem 1 with (1) and (2) justifies the name variational Bewley preferences.

4 Variational Bewley Preferences and Independence

Variational Bewley preferences do not obey the classical Independence axiom. We recall its definition from the Introduction

Independence. For all acts $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1)$, $f \succsim g \Leftrightarrow \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$.

As discussed in the previous section, it is known that variational Bewley preferences, in general, do not satisfy the Independence axiom. By Theorem 5 in Faro (2015), a variational Bewley preference that satisfies Independece is actually a Bewley preference with the form of (1). Something more can be said. In Faro (2015) it is shown that, if preferences are not transitive, then Dominance Independence does not imply Independence and vice-versa. For instance, the *justifiable preferences* axiomatized by Leherer and Teper (2011) are not transitive, and satisfy Independence but not Dominance Independence. Actually, under full transitivity, Dominance Independence and Independence are equivalent.

Consider now the following condition.

Weight Independence (WI). For all $f, g, h \in \mathcal{F}$ and for all $\alpha, \beta \in (0,1]$,

$$\alpha f + (1-\alpha)\,h \succsim \alpha g + (1-\alpha)\,h \Rightarrow \beta f + (1-\beta)\,h \succsim \beta g + (1-\beta)\,h.$$

The condition WI says that if an α-lottery between acts f and h is preferred to an α-lottery between acts g and h, than we can replace the mixing weight α with another mixing weight β and the preferences will not change. We will show now that WI is in fact equivalent to Independence.

Lemma 2 *Let $\succsim$ be a binary relation over $\mathcal{F}$. The following are equivalent:*

- *(i) $\succsim$ satisfies Independence.*
- *(ii) $\succsim$ satisfies WI.*
- *(iii) For all $f, g, h_1, h_2 \in \mathcal{F}$ and for all $\alpha, \beta \in (0,1]$,*

$$\alpha f + (1-\alpha)\,h_1 \succsim \alpha g + (1-\alpha)\,h_1 \Rightarrow \beta f + (1-\beta)\,h_2 \succsim \beta g + (1-\beta)\,h_2.$$

Proof. $(i) \Rightarrow (ii)$. Fix $f, g, h \in \mathcal{F}$ and $\alpha, \beta \in (0,1]$. Suppose $\alpha f + (1-\alpha)\,h \succsim \alpha g + (1-\alpha)\,h$, then by Independence $f \succsim g$, and again by Independence $\beta f + (1-\beta)\,h \succsim \beta g + (1-\beta)\,h$. Notice that the case with $\beta = 1$ is trivially true.

$(ii) \Rightarrow (i)$. Fix three acts $z_1, z_2, z_3 \in \mathcal{F}$ and a constant $\gamma \in (0,1)$.
Suppose $z_1 \succsim z_2$. Then taking $\alpha = 1$, $\beta = \gamma$ and $h = z_3$, WI implies $z_1 \succsim z_2 \Leftrightarrow 1z_1 + 0z_3 \succsim 1z_2 + 0z_3 \Rightarrow \gamma z_1 + (1-\gamma)z_3 \succsim \gamma z_2 + (1-\gamma)z_3$.
Suppose now $\gamma z_1 + (1-\gamma)z_3 \succsim \gamma z_2 + (1-\gamma)z_3$. Then taking $\beta = 1$, WI implies $1z_1 + 0z_3 \succsim 1z_2 + 0z_3 \Leftrightarrow z_1 \succsim z_2$.

$(ii) \Rightarrow (iii)$. Fix $f, g, h_1, h_2 \in \mathcal{F}$ and $\alpha, \beta \in (0,1]$ and suppose $\alpha f + (1-\alpha)\,h_1 \succsim \alpha g + (1-\alpha)\,h_1$. Using twice WI, $\alpha f + (1-\alpha)\,h_1 \succsim \alpha g + (1-\alpha)\,h_1 \Rightarrow 1f + 0h_1 \succsim 1g + 0h_1 \Leftrightarrow f \succsim g \Leftrightarrow 1f + 0h_2 \succsim 1g + 0h_2 \Rightarrow \beta f + (1-\beta)\,h_2 \succsim \beta g + (1-\beta)\,h_2$.

$(iii) \Rightarrow (ii)$ Obvious. ■

The equivalence between condition (i) and condition (iii) of Lemma 2 shows that the Independence axiom involves two types of independence: with respect to the mixing weights and the mixing acts. We note that this observation parallels MMR's discussion of their Axiom A.2 (see Lemma 1 in MMR), with the difference that they introduced a weak version of certainty independence of Gilboa and Schmeidler (1989) where the mixing is done only with respect to constant acts.

As we said in the previous section, Variational Bewley preferences do not satisfy Independence unless they are represented by (1), i.e. they are actually are actually Bewley preferences. Since Lemma 2 shows that Independence is actually equivalent to WI, it seems that a problematic feature of Independence for general Variational Bewley preferences is changing weights while mixing acts. Condition (*iii*) of Lemma 2 suggests therefore the following weakening of Independence.

Independence for Constant Weights (ICW). For all $f, g, h_1, h_2 \in \mathcal{F}$, and for all $\alpha \in (0, 1]$,

$$\alpha f + (1 - \alpha)h_1 \succsim \alpha g + (1 - \alpha)h_1 \Rightarrow \alpha f + (1 - \alpha)h_2 \succsim \alpha g + (1 - \alpha)h_2.$$

It is clear that ICW is a weakening of condition (*iii*) of Lemma 2 and therefore it is a weakening of Independence. ICW requires independence with respect to mixing with an arbitrary act h, provided that the mixing weights are kept constant. Given the discussion of Axiom A.5 (Dominance Independence), one would expect that Variational Bewley preferences satisfy ICW. This will be the case, as we show in the next proposition.

Proposition 3 *Any variational Bewley preference $\succsim$ satisfies ICW.*

Proof. Let $\succsim$ be a variational Bewley preference represented by a pair (u, η). By Faro (2015) we have that $B_0(\Sigma) = \{u(f) : f \in \mathcal{F}\}$. Define the binary relation $\trianglerighteq$ over $B_0(\Sigma)$ by

$$a \trianglerighteq b \Leftrightarrow f \succsim g \text{ for some } f, g \in \mathcal{F} \text{ such that } a = u(f),\ b = u(g).$$

By Faro 2015, $\trianglerighteq$ is well defined and moreover it satisfies *Reflexivity* ($a \trianglerighteq a$ for all $a \in B_0(\Sigma)$), *Mixture Continuity* (the set $\{\lambda \in [0, 1] : \lambda a + (1 - \lambda)b \trianglerighteq (\trianglelefteq) b\}$ is closed for all $a, b \in B_0(\Sigma)$) and *Affinity by Dominance* ($a \trianglerighteq b$ and $c \trianglerighteq d$ imply $\alpha a + (1 - \alpha)c \trianglerighteq \alpha b + (1 - \alpha)d$).

Claim. $\trianglerighteq$ satisfies *Affinity for Constant Weights*: for all $a, b, c_1, c_2 \in B_0(\Sigma)$ and all $\alpha \in (0, 1]$, $\alpha a + (1 - \alpha)c_1 \trianglerighteq \alpha b + (1 - \alpha)c_1 \Rightarrow \alpha a + (1 - \alpha)c_2 \trianglerighteq \alpha b + (1 - \alpha)c_2$.
Proof of the Claim. Notice that it is enough to show that $a \trianglerighteq b \Rightarrow c \trianglerighteq d$ whenever $a, b, c, d \in B_0(\Sigma)$ and $a - b = c - d$. Suppose that this is in fact the case and let $a, b, c_1, c_2 \in B_0(\Sigma)$ such that $\alpha a + (1 - \alpha)c_1 \trianglerighteq \alpha b + (1 - \alpha)c_1$. Notice that $(\alpha a + (1 - \alpha)c_1) - (\alpha b + (1 - \alpha)c_1) = (\alpha a + (1 - \alpha)c_2) - (\alpha b + (1 - \alpha)c_2)$, and therefore $\alpha a + (1 - \alpha)c_2 \trianglerighteq \alpha b + (1 - \alpha)c_2$.
Fix now $a, b, c, d \in B_0(\Sigma)$ and suppose $a \trianglerighteq b$.
• *Step 1.* Suppose there exists $\lambda \in (0, 1)$ such that $\lambda(a - b) = c - d$. Define : $e = \frac{1}{1-\lambda}(c - \lambda a)$. Clearly $e \in B_0(\Sigma)$. Then by Reflexivity $e \trianglerighteq e$ and by Affinity by Dominance $\lambda a + (1 - \lambda)e \trianglerighteq \lambda b + (1 - \lambda)e$. But $\lambda a + (1 - \lambda)e = c$ and $\lambda b + (1 - \lambda)e = \lambda b + c - \lambda a = \lambda(b - a) + c = d$, and therefore $c \trianglerighteq d$.
• *Step 2.* Assume now $a-b = c-d$ and note that $\forall \lambda \in (0, 1)$, $\lambda c+(1-\lambda)d-d = \lambda(c-d) = \lambda(a-b)$. Therefore by Step 1 $\forall \lambda \in (0, 1)$, $\lambda c + (1 - \lambda d) \trianglerighteq d$. Consider the set $A = \{\lambda \in [0, 1] : \lambda c + (1 - \lambda d) \trianglerighteq d\}$. A is closed by Mixture Continuity and hence the closure of A is equal to A. Hence since $(0, 1) \subseteq A$, and we must have $A = [0, 1]$. This implies $1 \in A$ and hence $c \trianglerighteq d$. This concludes the proof of the Claim.

Fix now $f, g, h_1, h_2 \in \mathcal{F}$ and $\alpha \in (0, 1]$ and suppose $\alpha f + (1-\alpha)h_1 \succsim \alpha g + (1-\alpha)h_1$. Let $a = u(f)$, $b = u(g)$, $c_1 = u(h_1)$, $c_2 = u(h_2)$ and define $k = u(\alpha f + (1-\alpha)h_1)$ and $j = u(\alpha g + (1-\alpha)h_1)$. We have $k \trianglerighteq j$ and moreover since $k(s) = u(\alpha f + (1-\alpha)h_1) = \alpha u(f(s)) + (1-\alpha)u(h_1(s))$ we obtain $\alpha a + (1-\alpha)c_1 \trianglerighteq \alpha b + (1-\alpha)c_1$. By Affinity for Constant Weights $\alpha a + (1-\alpha)c_2 \trianglerighteq \alpha b + (1-\alpha)c_2$. Therefore $\alpha u(f) + (1-\alpha)u(h_2) \trianglerighteq \alpha u(g) + (1-\alpha)u(h_2) \Leftrightarrow u(\alpha f + (1-\alpha)h_2) \trianglerighteq u(\alpha g + (1-\alpha)h_2)$ and finally $\alpha f + (1 - \alpha)h_2 \succsim \alpha g + (1 - \alpha)h_2$. ■

We conclude by noticing that in the proof of Proposition 3 the key axioms that imply ICW are Reflexivity, Mixture Continuity and Dominance Independence. In fact, the other axioms are used just to have a well defined binary relation $\trianglerighteq$ over $B_0(\Sigma)$.

5 Conclusion

David Schmeidler pionereed the stream of literature that studies generalizations of the Independence axiom. This axiom is a fundamental requirement if one wishes that agents behave according to the Expected Utility model. Relaxing Independence to Comonotoninc Independence and Certainty-Independence, Schmeidler was able to characterize the Choquet Expected Utility model and the MaxMin Expected Utility model. These two models are general enough to account for the preference pattern shown in the Ellsberg paradox.

This note follows his footpath. We study which notion of independence is related with the variational Bewley model developed in Faro (2015). We show that Independence is equivalent to a concept called Weight Independence. Weight Independence makes explicit the fact that the Independence axiom implies two type of independence: with respect to the act and with respect to the weights in the mixing. We proceed by studying a weakening of the Independence axiom, called Independence for Constant Weights. This property requires independence to hold only when the mixing weights are kept constant. Our main result shows that variational Bewley preferences satisfy Independence for Constant Weights.

References

[1] Anscombe. F.J. and R. Aumann (1963): *A definition of subjective probability*, **Annals of Mathematical Statistics** 34, 199-205.

[2] Cettolin, E. and A. Riedl (2013): *Revealed Preferences under Uncertainty: Incomplete Preferences and Preferences for Randomization.* **Journal of Economic Theory** 181, 547–585.

[3] Dunford, N. and J. T. Schwartz (1958): *Linear Operators, Part I: General Theory.* Wiley Classics Library Edition Published 1988.

[4] Ellsberg, D. (1961). *Risk, ambiguity and the Savage axioms.* **Quarterly Journal of Economics** 75, 643-669.

[5] Faro, J.H. (2015): *Variational Bewley preferences.* **Journal of Economic Theory** 157, 699-729.

[6] Fishburn, P. (1970): **Utility Theory for Decision Making**, New York: Wiley.

[7] Ghirardato, P., F. Maccheroni, M. Marinacci (2004): *Differentiating ambiguity and ambiguity attitude.* **Journal of Economic Theory** 118, 133-173.

[8] Gilboa, I. (2009): **Theory of Decision under Uncertainty**. Cambridge University Press.

[9] Gilboa, I. and D. Schmeidler (1989): *Maxmin expected utility with non-unique prior.* **Journal of Mathematical Economics** 18, 141-153.

[10] Hara, K., E.A. Ok and G. Riella (2019): *Coalitional Expected Multi-Utility Theory.* **Econometrica** 87, 933–980.

[11] Karni, E. and D. Schmeidler (1991): *Atemporal dynamic consistency and expected utility theory.* **Journal of Economic Theory** 54, 401–408.

[12] Lehrer, E. and R. Teper (2011): *Justifiable preferences.* **Journal of Economic Theory** 146, 762-774.

[13] Maccheroni, F., M. Marinacci and A. Rustichini (2006): *Ambiguity aversion, robustness and the variational representation of preferences.* **Econometrica** 74, 1447-1498.

[14] Schmeidler, D. (1989): *Subjective probability and expected utility theory without additivity.* **Econometrica** 57, 571-587.