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Aloisio Araujo
Alain Chateauneuf
José Heleno Faro

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Aloisio Araujo[a,b], Alain Chateauneuf[c,d] and José Heleno Faro[e]

[a]IMPA, Estrada Dona Castorina 110, 22.460-320, Rio de Janeiro, Brazil

[b]EPGE/FGV, Praia de Botafogo, 190, 22253-900, Rio de Janeiro, Brazil

[c]IPAG Business School, 184 bd Saint-Germain 75006 Paris Cedex 05, France

[d]Paris School of Economics, Université de Paris I, 106-112 Bld de l´Hopital, 75647 Paris Cedex 13, France

[e]Insper, Rua Quatá 300, Vila Olímpia 04546-042, São Paulo, Brazil.

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When the financial market has frictions there must be multiple or ambiguous risk-neutral probabilities. By providing a complete characterization of pricing rules of financial markets with a finite number of assets over a given state space, we are able to describe how we can recover an underlying financial market structure related to any finitely generated pricing rule. We provide a novel characterization for the set of efficient securities (that is, chosen by at least one rational expected utility agent) revealed by any valuation rule, which allows us to propose two meaningful notions of completeness: while a unique underlying *complete market* means that it is possible to replicate any position by trading efficient securities, the case of an underlying *efficient complete market* means that all financial positions are replicable and efficient. Our main result shows that efficient complete markets (with bid-ask spreads) is the prevalent case that emerges from the universe of all finitely generated pricing rules.

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Furthermore, any failure of efficient completeness requires valuation rules associate to sets of probabilities allowing disagreement about null events. In particular, the assumption of a Savagian state space almost precludes incompleteness. ***Journal of Economic Literature Classification Number*: D52, D53.**
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Introduction

Much of academic research and empirical analysis on valuation of financial assets has been done under the assumption of competitive and frictionless complete markets, *i.e.*, all agents act as price takers and can buy and sell all financial contracts over a given state space without paying any transaction cost. In such context, securities admit a perfect replication and the well-known linear pricing rule representation says that, by no-arbitrage, the price of any security can be computed by its expected value with respect to a unique risk-neutral probability. However, important evidences given by Hansen and Jagannathan (1991), Heaton and Lucas (1996), and Luttmer (1996), among others, says that the empirical implications of no arbitrage models are strongly affected by the presence of incompleteness and trading frictions like bid-ask spreads. Overall, the incorporation of trade imperfections in competitive securities market and its interplay with no-arbitrage is one of the most active research topics in economics and finance either theoretically or empirically. An important theoretical result in this line of investigation is that market imperfections generate pricing rules that are not linear but still compatible with the no-arbitrage assumption, as showed by Jouini and Kallal (1995). While in an arbitrage-free and frictionless complete financial market the corresponding pricing rules is linear, under the presence of incompleteness or frictions affecting tradeable securities, the no-arbitrage assumption leads to the existence of a set of risk-neutral probabilities, which can be interpreted as a kind of *ambiguity* about the stochastic discount factor. Furthermore, the multiple linear prices characterization of super-replication prices gives that any security has its price computed by the "largest" expected value based on the set of risk-neutral probabilities. We interpret this well-known result as a *caution valuation rule* based on the market information compatible with ambiguous risk-neutral probabilities. On the other hand, the axiomatic characterizations of pricing rules given by Chateauneuf, Kast and Lapied (1996, CKL for short), Jouini (2000), Castagnoli, Maccheroni and Marinacci (2002), among others, also find that the nonlinearity of pricing rules takes the special form of a maximum over a nonempty, closed and convex set of probabilities.

First, we provide a precise characterization of pricing rules of (arbitrage-free) financial markets[1] with a finite state space, and a finite number of tradeable assets. All the previous works characterizing pricing rules as mentioned above obtained a representation through a general nonempty, closed and convex set of probabilities. However, taking into account the perspective of empirical works where the set of attainable payoffs in a financial economy is generated by a finite set of assets (*e.g.*, Subsection 3.3. in Luttmer (1996)), it is not hard to see that the corresponding set of risk-neutral probability has its closure given by a polytope.[2] This observation clarifies that the previous works taking pricing rule as a primitive have characterized valuation rules that goes beyond the scope of financial markets with a finite number of assets. We fill this gaps by providing a condition over pricing rules equivalent to the polytope restriction, called *additivity over finite small markets*: there exists a finite collection of sub-market spaces (or *small markets*), where the bond is tradeable in any small market, the union of all small markets recover the entail market space, and the pricing rules is additive over each small market. We call all pricing rules under this characterization as a *finitely generated pricing rule*.

Once we have characterized the class of finitely generated pricing rule, we then provide an existence result that guarantee that all of these valuation rules are indeed a super-replication price of some arbitrage-free financial market. All those financial markets have in common the presence of a bond without bid-ask spreads but all the other financial positions are potentially associated to a bid-ask spread.

In Araujo, Chateauneuf and Faro (2012, ACF for short) we have shown that for a two-period financial market with a finite set of states of nature S, a pricing rule $C : \mathbb{R}^S \to \mathbb{R}$ reveals an underlying market given by a potentially incomplete financial market with only frictionless tradeable securities if, and only if, the set of *frictionless securities* F_C coincides with the set of *undominated securities* L_C. Being more precise, we have

$$F_C := \left\{ X \in \mathbb{R}^S : C(X) = -C(-X) \right\},$$

and

$$L_C := \left\{ X \in \mathbb{R}^S : Y > X \Rightarrow C(Y) > C(X) \right\}.$$

[1]Throughout the paper we impose no-arbitrage condition on the pricing rule, that is, on the underlyting financial market. Accordingly we avoid using the term "arbitrage-free" or "no-arbitrage" repeatedly.

[2]Recall that a set of probabilities $\mathcal{Q}$ is a polytope if there exist a finite number of probabilities $P_1, ..., P_k$ where $\mathcal{Q}$ is generated by all convex combination of such measures, *i.e.* $\mathcal{Q} = co\left(\{P_i\}_{i=1}^k\right)$, where co is the convex hull operator.

In this paper we show that undominated securities can be identified with the set of *efficient securities*. Efficiency of a security means that at least one *rational expected utility agent* chooses such position in the financial market. This notion has been studied in the literature since Dybvig's (1988a, 1988b) characterization of efficient trading strategies in the context of frictionless complete markets. The Dybvig's seminal result shows that a position X is efficient if, and only if, it provides at least as much *net* payoff in cheaper states of nature according the unique risk neutral probability.[3] In another seminal contribution, Jouini and Kallal (2001) shows that, even the Dybvig's distributional price notion is not relevant anymore, the efficiency of a position X is equivalent to the existence of a strictly positive linear pricing rule P providing the valuation of X and it gives the right to at least as much *net* payoffs in cheaper states of nature according P.

Building on the characterization of L_C as the set of efficient securities, we introduce two meaningful notions of completeness associated to a pricing rule.[4] We say that C *is a pricing rule of a complete market of securities generated by efficient assets* when $Span\,(L_C) = \mathbb{R}^S$.[5] Here, the idea is that the pricing rules reveals a collection of efficient securities that makes all contingent claim replicable. Moreover, in order to capture the context where the marketed space contains only efficient securities, we say that C *is a pricing rule of an efficient complete market of securities* when $L_C = \mathbb{R}^S$.[6] To summarize, the fundamental difference between these notions of completeness is that, although in both cases all financial positions are tradeable by trading efficient securities, only in the case of an efficient complete market every security can be used in a rational way.[7]

Next, one of our contribution is to provide a constructive way for finding an underlying market for any finitely generated pricing rule C. By showing that the set of undominated securities L_C can be decomposed in a finite union of (polyhedral) convex cones, we get that the the family of securities characterizing the underlying market emerges by invoking the well-known Minkowski-Weyl's theorem.

[3] See also Peleg and Yaari (1975) and Dybvig and Ross (1982, 1986).

[4] It is worth noticing that, in general, L_C is a cone but not necessarily convex.

[5] This is consistent with the fact (see Proposition 13) that $Span\,(L_C)$ is proved to be the smallest market space generated by C.

[6] This is consistent with the fact in this case $Span\,(L_C) = L_C = \mathbb{R}^S$ is the unique market space generated by C. Theorem 8 shows, in special, that this is equivalent to the fact the closed and convex set of priors characterizing C contains only strictly positive probabilities.

[7] An interesting weaker notion called "effectiveness" was proposed by Baccara et al. (2006). Actually, this notion is equivalent to the notion of "zero inefficiency cost" proposed by Jouini and Kallal (2001). For details, see the discussion in Baccara et al. (2006), p. 67.

Our main contribution in this paper shows that, in general, *finitely generated pricing rules reveals an efficient complete market of securities*. Indeed, we show that $L_C = \mathbb{R}^S$ holds if, and only if, the set of probabilities Q characterizing the pricing rule C contains only strictly positive probabilities. Since any set of probabilities can be approximated by polytopes generated by strictly positive probabilities,[8] we can view the case of efficiently complete market as the prevalent case of financial markets revealed by pricing rules. Furthermore, we show that efficient complete markets is equivalent to a property of pricing rules adapted from Kreps (1979) and Epstein and Marinacci (2007) saying that for all $X, Y, Z \in \mathbb{R}^S$,

$$C\left(X \wedge Z\right) > C\left(X \wedge Y \wedge Z\right) \Rightarrow C\left(X\right) > C\left(X \wedge Y\right),$$

and in this case we say that C satisfy the KEM property.

Below, we discuss an example that illustrates substantial part of our contribution.

The Ellsberg securities market

Consider a market where in principle any conceivable security or financial position is tradeable. The special feature in this example is that the information about future contingencies follows the famous Ellsberg urn likelihood structure. There is an urn that contains 30 red balls and 60 other balls that are either green or blue. Hence, the state space is given by $S = \{r, g, b\}$ and a contingent claim X is a function from S to $\mathbb{R}$, denoted by $X = (x_1, x_2, x_3)$. Given that the probability of the state r is *unambiguous*, let us to assume that the Arrow security $\{r\}^* := (1, 0, 0)$ is efficient and frictionless, which allows us to consider that, w.l.o.g., its risk neutral valuation reveals a price is given by $1/3$. On the other hand, due to the ambiguity concerning the likelihood of the states g and b, we aim to study how different risk neutral valuations concerning such states can reveal different properties of the underlying market structure. In this way, we consider all pricing rules $C : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that the valuation of the Arrow security related to state r is given by $C\left(1, 0, 0\right) = 1/3$. In the case of an extreme ambiguous risk-neutral probability, any distribution of the type $\left(1/3, \beta, \frac{2}{3} - \beta\right)$ are considered.[9] In this case, the pricing rule C is given by

$$C\left(X\right) = \max\left\{\frac{1}{3}x_1 + \beta x_2 + \left(\frac{2}{3} - \beta\right)x_3 : \beta \in \left[0, \frac{2}{3}\right]\right\}.$$

[8]Here, we are taking into account the well known result about the density of polytopes under the Hausdorff metric.

[9]Note that we can consider the closure of the considered set of risk-neutral probabilities.

From the main result in ACF (2012), since we have that

$$L_C = F_C = \left\{x \in \mathbb{R}^3 : x_2 = x_3\right\},$$

it follows that the underlying market is an incomplete market without bid-ask spreads. Actually, the underlying market can be described by the following collection of assets and prices[10]

$$\mathcal{M} = \{ \ (\{r\}^*, 1/3), (\{g, b\}^*, 2/3)\}.$$

In order to illustrate some of our results discussed above, consider the more general case of pricing rules generated by the pairs give by $\delta = (\delta_1, \delta_2) \in [0, 1/3] \times [0, 1/3]$,

$$\mathcal{Q}_\delta := \{(1/3, \beta, 2/3 - \beta) : \beta \in [\delta_1, 2/3 - \delta_2]\}.$$

Note that the case where $\delta_1 = \delta_2 = 0$ gives the previous case of incomplete markets, while if $\delta_1 = \delta_2 = 1/3$ then we obtain the pricing rule

$$C(X) = \frac{1}{3}(x_1 + x_2 + x_3)$$

where $F_C = L_C = \mathbb{R}^3$, which reveals the frictionless efficient complete market

$$\mathcal{M} = \{(\{r\}^*, 1/3); \ (\{g\}^*, 1/3); \ (\{b\}^*, 1/3)\}.$$

When $\delta_1, \delta_2 \in (0, 1/3)$ the pricing rule is given by

$$C(X) = \max\left\{\frac{1}{3}x_1 + \beta x_2 + \left(\frac{2}{3} - \beta\right)x_3 : \beta \in [\delta_1, 2/3 - \delta_2]\right\}.$$

Moreover, we obtain a unique underlying market of consisting all Arrow securities and of the frictionless bond (with price one), where the pairs of bid-ask states prices $\left(q_s^A, q_s^B\right)_{s \in S}$ satisfy[11]

$$q_r^A = 1/3 = q_r^B; \ q_g^A = 2/3 - \delta_2; q_b^A = 2/3 - \delta_1; \ q_g^B = \delta_1; \ q_b^B = \delta_2.$$

Which indeed is complete, but can be proved to be efficient (*i.e.,* $L_C = \mathbb{R}^3$) in the prevalent case that is for every $(\delta_1, \delta_2) \in (0, 1/3) \times (0, 1/3)$.

[10] Given a subset $E \subset S$, E^* denotes the $\{0, 1\}$-valued function such that $E^*(s) = 1$ iff $s \in E$.

[11] We note that follows that the bid-ask price of other bets are given by

$$\begin{aligned} q_{rg}^A &= 1 - \delta_2; \ q_{rb}^A = 1 - \delta_1; \ q_{gb}^A = 2/3; \\ q_{rg}^B &= 1/3 + \delta_1; \ q_{rb}^B = 1/3 + \delta_2; \ q_{gb}^B = 2/3. \end{aligned}$$

Also, it is worth noting that the case of complete market where efficiency fails to occurs in the case where either $\delta_1 = 0$ and $\delta_2 > 0$ obtaining

$$C(X) = \max_{\beta \in [0, 2/3 - \delta_2]} \left\{ \frac{1}{3} x_1 + \beta x_2 + \left(\frac{2}{3} - \beta \right) x_3 \right\} \text{ and, for instance, } \{r, b\}^* \notin L_C,$$

or $\delta_1 > 0$ and $\delta_2 = 0$ obtaining

$$C(X) = \max_{\beta \in [\delta_1, 2/3]} \left\{ \frac{1}{3} x_1 + \beta x_2 + \left(\frac{2}{3} - \beta \right) x_3 \right\} \text{ and, for instance, } \{r, g\}^* \notin L_C.$$

To summarize, the prevalent case is given by the collection of all pairs $(\delta_1, \delta_2) \in (0, 1] \times (0, 1]$ generating efficient complete markets. Otherwise, any failure of efficient completeness requires valuation rules represented by sets of probabilities measures allowing disagreement on null events.

Incomplete market models: A caveat

The previous example with an Ellsbergian securities market illustrates the fact that incomplete markets occurs in a Savagean context (*i.e.,* any event is foreseen) only if some probability consistent with the market gives probability zero to some event, which constitutes one of our main messages in this paper. Keynes (1936, ch. 16) argued that the limited ability of agents to cope with uncertainty leads to missing markets because they are reluctant to make more than limited contractual commitments into the future and as a consequence some markets are missing. Our result indicates that Keynes' theory of missing markets cannot be well accommodate as a robust phenomena by the Savagian state space approach. Indeed, one immediate consequence of our result concerning the prevalence of efficient complete markets is that the case of incompleteness is related to the context where market data must be consistent with a disagreement about zero probabilities events. In this way, models with endogenous state space or unforeseen contingencies might be more successful in obtaining incompleteness of financial markets as the rule rather than the exception. Also, while some works only relates the possibility of multiplicity of risk-neutral probabilities to incomplete markets, our work shows that the case of ambiguous risk-neutral probabilities is in fact related to efficient complete markets with bid-ask spreads. In this way, a natural question that arises is how methods for pricing a new non-redundant derivative security, like the one proposed by Boyle, Feng, Tian, and Wang

(2008) for incomplete markets without bid-ask spreads, can be applied to the case of financial markets with multiple risk-neutral probabilities but with efficient completeness.

Our result concerning incompleteness is consistent with the relatively weak link obtained in the literature between uncertainty aversion and incomplete markets. Recall that the well-known inertia interval obtained by Dow and Werlang (1992) presents a statement on the optimal portfolio choice corresponding to exogenously determined prices for a given initial sure position. In their study about uncertainty aversion and incomplete markets, Mukerji and Tallon (2001) observed that it does not follows from Dow and Werlang (1992) that no-trade constitute the solution in a general equilibrium context. Actually it is simple to see in an Edgeworth box that, in general, the area of mutually advantageous trade is nonempty for uncertainty averse agents (given by mild convex preferences).[12] Indeed, no-trade is an equilibrium outcome for such class of economies if, and only if, endowment is Pareto optimal to begin with. The introduction of ambiguity aversion in an economy through Choquet functionals or more general ones forms of uncertainty aversion, in general, does not precludes the trade in risk sharing contracts and would not be a reason for incomplete risk sharing.[13]

Mukerji and Tallon (2001) also studies if uncertainty aversion in a heterogeneous agent CEU model might lead to an endogenous breakdown in markets for some assets. Following the previous statement concerning Edgeworth box economies, Mukerji and Tallon (2001) observed that more conditions must be imposed. They showed that a sufficient condition is the introduction of a idiosyncratic components, *i.e.,* a component in asset payoffs that is independent of the endowments in the economy and the payoff of any other asset as well. Under idiosyncratic components, they showed that "when the assets available to trade risk among agents are affected by idiosyncratic risk, and if agents perceive this idiosyncratic component as being ambiguous and the ambiguity is high enough, then every equilibrium involves no trade over these assets".[14] Hence, Mukerji and Tallon (2001) shows how ambiguity aversion may endogenously limit the scope of risk sharing obtainable through the bonds traded in an economy. However, this conclusion can be viewed also as a negative result because they impose the strong condition of idiosyncratic components. Indeed, Rinaldi (2009) showed that Mukerji and Tallon's

[12] See also the discussion provided by Ghirardato and Siniscalchi (2012) on this topic that takes into account a broarder class of preferences.

[13] For example, Chateauneuf, Dana and Tallon (2000) proved, under common convex capacity, that risk sharing proceeds just as in an economy with SEU agents.

[14] Note that this is to be contrasted with the situation in which agents are SEU, in which standard replication and diversification arguments ensure that full risk sharing may be obtained and the equilibrium is Pareto optimal, *e.g.*, see Werner (1997).

result cannot hold for the class of smooth variational preferences of Maccheroni, Marinacci and Rustichini (2006). Finally, Rigotti and Shannon (2012) shows that if ambiguity is modeled using variational preferences then indeterminacies and no-trade are not the typical equilibrium result. This results contrasts sharply with the conclusion about the possibility of endogenous incomplete markets when agents have incomplete preferences *a la* Bewley (2002), as Rigotti and Shannon (2005) have shown.

Our Main Result and the Empirical Literature

We fully characterize pricing rules of finite financial markets and show that efficient complete markets allowing, in general, bid-ask spreads is the prevalent case. Also incompleteness of financial markets are revealed by the case of an ambiguous valuation that fails to satisfy the fully agreement over null events (or, the mutually absolutely continuity condition). We conclude this Introduction with a briefly discussion of some important result from the empirical literature taking into account our findings.

Recall that in the case of frictionless market completeness, a central result in asset pricing says that the stochastic discount factor can be identified with the intertemporal marginal rates of substitution of a representative agent.[15] Indeed, an usual empirical strategy is to connect intertemporal marginal rates of substitution to stochastic discount factors. An important methodological constraint to the econometric evaluation of asset pricing models is that, for a given financial data set, the typical case is the existence of multiple stochastic discount factors. As well highlighted by Hansen, Heaton, Lee, and Roussanov (2007),

> "only when the econometrician uses a complete set of security market payoffs will there be a unique discount factor."

[15] See, for instance, LeRoy (1973), Lucas (1978), and Harrison and Kreps (1979). Also, given a prior objective probability P^* over the state space, recall that each stochastic discount factor corresponds to a risk-neutral probability and vice-versa. Indeed, given a stochastic discount factor $d : S \to \mathbb{R}$ the corresponding risk-neutral probability P follows by taking, for all s', $P(s') := d_s P^*(s') / E_{P^*}(d)$, which gives that for all $X \in \mathbb{R}^S$

$$E_P(X) = E_{P^*}(dX),$$

where dX denotes the mapping defined by $(dX)(s) := d(s) X(s)$ for all $s \in S$. On the other hand, given a set of risk-neutral probabilities $\mathcal{Q}$, the induced set of stochastic discount factor is given by

$$D_{P^*} := \left\{ \begin{array}{c} d \in \mathbb{R}^S : \text{ for some } P \in \mathcal{Q}, \\ E_{P^*}(dX) = E_P(X), \text{ for all } X \in \mathbb{R}^S \end{array} \right\}.$$

This observation is consistent with the fact that either an explicit economic model is assumed that generated a discount factor, or an *ad hoc* identification method should be taken *a priori* in order to find a discount factor. In this way, dynamic macroeconomic equilibrium models have been used in order to produce candidate discount factors. For instance, under incompleteness, Hansen and Jagannathan (1991) suggested that stochastic discount factors have to be much more volatile than the IMRSs of typical representative agent models. On the other hand, when data on transaction costs are taken into account, estimates presented in Luttmer (1996) indicate that the low variability of these IMRSs is consistent with asset returns indicating a smallest volatility of the stochastic discounting factors. The dichotomy above between Hansen and Jagannathan (1991) and Luttmer (1996) is consistent with our results: Incompleteness leads to a relatively large ambiguity about the stochastic discount factors while transactions costs are consistent with relatively mild levels of ambiguous stochastic discount factors.

Pricing Rules of Finite Financial Markets

We consider a single-period economy where the uncertainty is modeled by a finite state space $S = \{s_1, ..., s_n\}$. Let Δ be the set of all probability measures on $(S, 2^S)$. Also, Δ^+ denotes the set of strictly positive probabilities or the set of probability measures with full support:

$$Supp\,[P] := \{s : P\,(s) > 0\} = S.$$

Also, given a subset $\mathcal{Q} \subset \Delta$ we denote $\mathcal{Q}^+ := \mathcal{Q} \cap \Delta^+$ and $\mathcal{Q}^\partial = \mathcal{Q} \cap (\Delta^+)^c$.

A mapping $X : S \to \mathbb{R}$ is a security that gives the right to $X\,(s)$ units of consumption or wealth in the second period in each state of nature $s \in S$. A bet on the event A is given by the security $A^* : S \to \{0, 1\}$, where $A^*\,(s) = 1$ iff $s \in A$.

Following CKL (1996), Jouini (2000), Jouini and Kallal (2001), Castagnoli, Maccheroni and Marinacci (2002), ACF (2012) we have the definition:[16]

Definition 1 *A mapping $C : \mathbb{R}^S \to \mathbb{R}$ is a pricing rule if*

i) C is sublinear, $i.e.$,

$$C\,(\lambda X) = \lambda C\,(X)\,,\ \ (\textit{positive homogeneous})\,,\ \textit{and}$$

$$C\,(X + Y) \leq C\,(X) + C\,(Y)\,,\ \ (\textit{subadditive})\,,$$

[16]The intuition of each property can be find in any of the previous quoted papers.

for all $X, Y \in \mathbb{R}^S$ and all non-negative real number λ;

ii) C is arbitrage free, $i.e.$, $C(X) > 0$ for any nonzero security $X \geq 0$;

iii) C is normalized, $i.e.$, $C(S^*) = 1$;

iv) C is monotonic, $i.e.$, $C(X) \geq C(Y)$ for all $X, Y \in \mathbb{R}^S$ s.t. $X \geq Y$;

v) C is constant additive, $i.e.$, $C(X + kS^*) = C(X) + k$, for all $X \in \mathbb{R}^S$ and all real number k.

It seems worth noticing that subadditivity is the property that captures the notion of a caution valuation. It says that, in general, it is less expensive to purchase a portfolio of securities than to purchase each security separately. Actually, the next well-know result can be viewed as a valuation rule that captures the willingness to accept of a maxmin ambiguity averse[17] and risk neutral seller:

Theorem 2 *Given a pricing rule $C : \mathbb{R}^S \to \mathbb{R}$, there exists a unique closed and convex set $\mathcal{Q}$ of probability measures, where at least one element is strictly positive, such that for any security X*

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X)$$

It seems important to take a caveat at this point. Usually, the set of attainable payoffs in a financial economy is generate by a finite set of assets (*e.g.*, Subsection 3.3. in Luttmer (1996)). Hence, the corresponding set of risk-neutral probability has its closure given by a polytope by a well know result (e.g., Theorem 2.4.6 in Schneider (1993)). For instance, assume that $\#S = 3$, for all $\varepsilon > 0$ such that $\mathcal{Q} = B\left(\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \varepsilon\right) \subset \Delta$, the corresponding pricing rule cannot be related to a finite financial market.

We call C a finitely generated pricing rule if C is a pricing rule with the additional property that the set of probabilities Q that characterizes C is given by a polytope. Next, we identify the property that characterizes finitely generated pricing rules.

[17] Maxmin ambiguity averse agents *a la* Gilboa and Schmeidler (1989) are characterized by a set of priors $\mathcal{C}$ and a utility index $u : \mathbb{R} \to \mathbb{R}$ where utility functions can be written by $U : \mathbb{R} \to \mathbb{R}$ with

$$U(X) = \min_{P \in \mathcal{C}} E_P(u(X)).$$

Also, risk neutrality is captured by the fact that $u(x) = x$ for all $x \in \mathbb{R}$. The willingness to pay is given by $U(X)$ and the willingness to accept is given by $-U(-X)$. An alternative approach to maxmin expected utility model have been provided by Gilboa, Maccheroni, Marinacci, and Schmeidler (2010).

Definition 3 *A mapping* $C : \mathbb{R}^S \to \mathbb{R}$ *is* additive over finite small markets *if there exist a finite collection of polyhedral cones* $\{W_l\}_{l=1}^{L}$ *with the property that* $\cup_l W_l = \mathbb{R}^S$*,* $span\,\{S^*\} \subset \cap_l W_l$*, and given an index* $l \in \{1, ..., L\}$ *for all* $X, Y \in W_l$

$$C\,(X + Y) = C\,(X) + C\,(Y)\,.$$

This property says that pricing rules should be additive over a finite collection of suitable regions of the marketed space. We might view each region where additivity holds as a particular "small market" in which a unique fair linear price works. The notion of small markets can be viewed in a similar way of 'small words' discussed by Savage (1954), and further analyzed by Chew and Sagi (2006, 2008). This suggests that each small market can be associated with a distinct restricted domain of contracts related to a sufficiently simple source of uncertainty, which can be referred as an event called 'small world'. For instance, a pricing rule can be additive within bets over sub-events of a small world event, but equally likely complementary events in another small world may not pricing in the same way. Therefore, it might be less expensive to purchase a portfolio of bets on different small worlds than to purchase each of those bets separately.

The next result shows that the previous condition is the essence of a polytope set of linear pricing rules that emerges from the market data.

Theorem 4 *Given a pricing rule* $C : \mathbb{R}^S \to \mathbb{R}$ *, the corresponding set of probabilities* $\mathcal{Q}$ *is a polytope if, and only if,* C *is additive over finite small markets.*

The following result is very important because it allows us to say that a finitely generated pricing rule C is a pricing rule of a finite (arbitrage-free) market of securities.[18] In another way, the polytope condition is not only necessary but also a sufficient condition for C being a pricing rule of a finite financial market.

Theorem 5 *The mapping* C *is a super-replication price of some arbitrage-free financial market* $\mathcal{M} = \left\{X_j, \left(q_j^A, q_j^B\right)\right\}_{j=0}^{m}$ *if, and only if, the mapping* C *is a finitely generated pricing rule.*

One important consequence our Theorem 5 is that finitely generated pricing rules can be always associated to incomplete markets or bid-ask spreads. Later, we will provide a constructive result about how any finitely generated pricing rules can reveal an underlying financial markets.

[18]See Appendix Part A for the definition and basic facts about finite financial market of securities without arbitrage opportunities.

Efficiency and Market Completeness

Recall that given a pricing rule $C : \mathbb{R}^S \to \mathbb{R}$, the induced set of frictionless securities is defined by

$$F_C := \left\{ X \in \mathbb{R}^S : C(X) + C(-X) = 0 \right\}.$$

Clearly, the fact that $X \in F_C$ means that the security X can be bought and sold without any frictions. Any pricing rule C gives rise to a linear subspace of frictionless securities F_C.[19]

Given a pricing rule C, the induced set of undominated securities is defined by

$$L_C := \left\{ X \in \mathbb{R}^S : Y > X \Rightarrow C(Y) > C(X) \right\}.$$

An undominated security X is characterized by the property saying that if some of its contingent payoff is replaced by a higher payoff, then the resulting security is strictly more expensive than the original one. As observed in ACF (2012), note that for all pricing rules C it follows that any frictionless security X is undominated, *i.e.,* $F_C \subset L_C$. The set of undominated securities L_C plays a fundamental role in our study.

Given a finitely generated pricing rule, we show next that the revealed set of undominated securities L_C can be viewed as the set of efficient securities. Dybvig (1988a, 1988b) introduced the notion of efficient securities in the following way: Given a uncertain endowment $X_0 \in \mathbb{R}^S$ and a pricing rule C, a security X is efficient with respect to the pair (C, X^0) if there exists a risk averse von Neumann-Morgenstern rational agent endowed with some initial wealth X^0 for which X is an optimal choice given the pricing rule C. Denoting by $Eff_{X^0}(C)$ the set of efficient securities induced by the pair (C, X^0), Jouini and Kallal (2001) provides an important characterization of the set of efficient securities

$$Eff_{X^0}(C) = \left\{ \begin{array}{c} X \in \mathbb{R}^S : C(X) = E_P(X) \text{ for some } P \in \mathcal{Q}^+ \\ \text{and } X(s) + X^0(s) > X(s') + X^0(s') \Rightarrow P(s) \leq P(s') \end{array} \right\}.$$

The intuition behind this characterization of efficient securities is quite natural. Given a pricing rule C, the set $Eff_{X^0}(C)$ captures the collections of all securities evaluated through some risk neutral probability $P \in \mathcal{Q}^+$ with the additional property that larger payoffs are related to cheaper state prices induced by P.[20]

The connection between L_C and $Eff_{X^0}(C)$ follows as:

[19] See Lemma 3 of ACF (2012).

[20] This approach assumes that rational agents considered in the primitive definition of efficient securities only care about the distribution of payoffs because all the states of nature are equiprobable for their beliefs.

Theorem 6 *For any endowment $X^0 \in \mathbb{R}^S$, we have that $Eff_{X^0}(C) \subset L_C$. Also, given $X \in L_C$, there exists $X^0 \in \mathbb{R}^S$ such that $X \in Eff_{X^0}(C)$. In another way,*

$$L_C = \bigcup_{X^0 \in \mathbb{R}^S} Eff_{X^0}(C).$$

Building on the previous characterization, we aim to provide an important distinction between two notions capturing completeness of financial markets. The first notion is a strong condition saying that any financial position X is undominated, that is,

$$L_C = \mathbb{R}^S.$$

In this case we say that C is a pricing rule of an "efficient complete market".

The second notion of completeness is less demanding. It just says that any financial position can be generated by some portfolio that takes into account only undominated securities, that is,

$$Span\,(L_C) = \mathbb{R}^S.$$

In this case, we say that C is a pricing rule of a "complete market of securities generated by efficient assets".

In order to illustrate the main difference between both notions of complete markets, let us consider the following

Example 7 *There are two states of nature and assume that the pricing rule is given by*

$$C(x_1, x_2) = \max_{\alpha \in [1/2, 1]} \{\alpha x_1 + (1-\alpha) x_2\}.$$

We note that in this case the set of undominated securities is the convex cone given by

$$L_C = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq x_1\right\} = cone\,\{S^*, -S^*, \{s_2\}^*\}$$

Also, the set of frictionless securities satisfies $F_C = span\,\{S^*\}$. We note that $\mathcal{M}$ is a complete market but not an efficient complete market of securities. Actually, it is simple to show that C is the super-replication price induced by the market of securities given by[21]

$$\mathcal{M} = \{(S^*, 1)\,; (\{s_2\}^*, (0, 1/2))\}.$$

[21] Note that the case of an Arrow security $\{s_i\}^*$ with bid-price $q_s^B = 0$ can be reinterpreted as the case of an Arrow security with a unique price $q_s = q_s^A$ in which we have short-sales constraint concerning such security.

Next, we present a result with very important consequences:

Theorem 8 *The following conditions are equivalent:*
(i) *C is a pricing rule of a efficient complete market of securities.*
(ii)*The set of probabilities measures representing C contains only full support probabilities.*
(iii) *(KEM property) For all $X, Y, Z \in \mathbb{R}^S$,*

$$C(X \wedge Z) > C(X \wedge Y \wedge Z) \Rightarrow C(X) > C(X \wedge Y).$$

One consequence of this result is that among the universe of all pricing rules, the prevalent case is the class of pricing rules of efficient complete markets with bid-ask spreads.[22] We do not mean that real financial markets characterized by a system of contracts which involve only a limited commitments into the future are the exception rather than the rule. Our message is that, in a financial market with unforeseen contingencies, incompleteness is always a consequence of an extreme form of ambiguity concerning linear valuation rule in the sense that the extended set of risk-neutral probabilities consistent with the market should allows for a disagreement about some zero probability event. Finally, the condition (iii) says that efficient complete markets is equivalent to a property of pricing rules that we adapted from Kreps (1979) and Epstein and Marinacci (2007), called KEM property. This property provides a useful way for testing if a given pricing rule is related to a efficient complete market of securities. For instance, in our previous example with two states of nature, by taking $X = (1, 1)$, $Y = (1, 0)$ and $Z = (-1/2, 1)$ we get that the KEM property is false.

One first consequence of our Theorem 8 is that incomplete markets implies in a very special property of the set of linear valuation rules consistent with the market:

Corollary 9 *If C is a pricing rule related to an incomplete market of securities then there exists a linear pricing rule with non full-support in $\mathcal{Q}$,* i.e.*, $\mathcal{Q}^{\partial} \neq \emptyset$. Also, if a security X is not tradeable we have that if $C(X) = E_P(X)$ with $P \in \mathcal{Q}$ then $P \in \mathcal{Q}^{\partial}$.*

Recall that one of the most well-known result concerning financial markets without frictions on tradeable securities says that the case of complete markets is equivalent to the case of a unique (strictly positive) risk-neutral probability. Under the possibility of frictions over tradeable securities our Theorem 8 says that completeness is captured by the fact that all linear

[22] We note that the class of all polytope is dense when we consider the metric space of all convex bodies endowed with the Hausdorff metric (See, for instance, Schneider (1993)).

pricing valuation consistent with the market is strictly positive. The next result is an alternative way of rewritten this characterization of efficient complete markets:

Corollary 10 *C is finitely generated pricing rule of an arbitrage-free efficient complete market of securities $\mathcal{M} = \left\{X_j, \left(q_j^A, q_j^B\right)\right\}_{j=0}^m$ if, and only if, there exists a finite set of risk neutral probabilities $\{P_i\}_{i=1}^n \subset \Delta^+$ such that*

$$C(X) = \max_{1 \le i \le n} E_{P_i}(X), \text{ for all } X \in \mathbb{R}^S.$$

*Moreover, the market is frictionless (*i.e. $F_C = L_C$*) if, and only if, there is only one risk-neutral probability.*

Revealing Financial Markets from Finitely Generated Pricing Rule

Next, we present one of our main result:

Theorem 11 *If C is a finitely generated pricing rule then C is a super-replication price of an arbitrage-free financial market*

$$\mathcal{M} = \left\{X_j, \left(q_j^A, q_j^B\right)\right\}_{j=0}^m,$$

where $Span\left(\{X_j\}_{j=0}^m\right) = Span(L_C)$. Moreover, the efficient market space L_C can be rewritten as a finite union of convex cones, where each one is generated by subset of securities $X_j's$. Finally, the pricing rule C is additive over each of such convex cones.

The next example shows how our constructive proof can be useful (see the proof in the Appendix B).

Example 12 *Consider the case of three states of nature $S = \{s_1, s_2, s_3\}$ and a finitely generated pricing rule related to the set of priors given by*

$$\mathcal{Q} =: co(\{P_1, P_2, P_3\}),$$

where $P_i = \frac{1}{2}(S^* - \{s_i\}^*)$. In this case it is easy to see that,[23] *given $X = (x_1, x_2, x_3) \in \mathbb{R}^3$,*

$$C(X) = \frac{1}{2}\max\{x_1 + x_2,\ x_1 + x_3,\ x_2 + x_3\},$$

[23] Note that

$$\mathcal{Q} = co(\{P_1, P_2, P_3\}) = \{(p_1, p_2, p_3) \in \Delta : 0 \le p_i \le 1/2 \text{ for all } i\}.$$

and $F_C = Span\,(S^*)$. *Also, it is not hard to show that*

$$L_C = \left\{X \in \mathbb{R}^3 : \#\left\{i : x_i = r \text{ for some } r \in \mathbb{R}\right\} \geq 2\right\}.$$

and hence, $Span\,(L_C) = \mathbb{R}^3$, *that is, any underlying market is a complete market generated by efficient assets but this financial market is not an efficient complete market of securities. For instance, we can take*

$$\begin{aligned} S^*,\ q_0 &= 1 \\ \{s_i\}^*,\ q_i^A &:= 1/2 > 0 =: q_i^B,\ (i = 1, 2, 3), \end{aligned}$$

which has its superhedging price given by C. *We note that*

$$\begin{aligned} V_1 &= \{X : x_3 \geq x_1 \text{ and } x_2 \geq x_1\} \\ V_2 &= \{X : x_3 \geq x_2 \text{ and } x_1 \geq x_2\} \\ V_3 &= \{X : x_2 \geq x_3 \text{ and } x_1 \geq x_3\}. \end{aligned}$$

Also, considering

$$\mathcal{J}_{i_0} := \left\{J \subset \{1, 2, 3\} : i_0 \in J \text{ and } co\left(\{P_j\}_j\right) \cap \Delta^+ \neq \emptyset\right\},$$

and for all $J \in \mathcal{J}_{i_0}$

$$V_{i_0}^J = \left\{X \in V_{i_0} : E_{P_{i_0}}(X) = E_{P_j}(X) \text{ for all } j \in J\right\},$$

we obtain

$$\begin{aligned} V_1 \cap L_C &= \{X : x_3 = x_1 \text{ and } x_2 \geq x_1\} \cup \{X : x_3 \geq x_1 \text{ and } x_2 = x_1\} \\ &= V_1^{\{1,2\}} \cup V_1^{\{1,3\}} \\ V_2 \cap L_C &= \{X : x_3 = x_2 \text{ and } x_1 \geq x_2\} \cup \{X : x_3 \geq x_2 \text{ and } x_1 = x_2\} \\ &= V_2^{\{1,2\}} \cup V_2^{\{2,3\}} \\ V_3 \cap L_C &= \{X : x_2 = x_3 \text{ and } x_1 \geq x_3\} \cup \{X : x_2 \geq x_3 \text{ and } x_1 = x_3\} \\ &= V_3^{\{1,3\}} \cup V_3^{\{2,3\}} \end{aligned}$$

and it easy to see that there are only three different convex cones (for instance, $V_1^{\{1,2\}} = V_3^{\{2,3\}}$*):*

$$\begin{aligned} L_C &= \{X :\ x_2 \geq x_1 = x_3\} \cup \{X : x_3 \geq x_1 = x_2\} \cup \{X : x_1 \geq x_2 = x_3\} \\ &= cone\left(\{S^*, -S^*, (0,1,0)\}\right) \cup cone\left(\{S^*, -S^*, (0,0,1)\}\right) \cup cone\left(\{S^*, -S^*, (1,0,0)\}\right) \end{aligned}$$

That is, L_C *is the union of three convex cones.*

One important step in the proof of Theorem 11 is that, given a pricing rule $C : \mathbb{R}^S \to \mathbb{R}$, L_C can be written as a finite union of convex cones. In special, when the set of probability measures $\mathcal{Q} = co\left(\{P_i\}_{i=1}^k\right)$ related to C is such that $P_i \in \Delta^+$ for all $i \in \{1, ..., k\}$ it follows that

$$L_C = \bigcup_{1 \leq i \leq k} V_i,$$

where V_i is the convex cone given by the set $\left\{X \in \mathbb{R}^S : C(X) = E_{P_i}(X)\right\}$, which is a polyhedral set. By the Minkowski-Weyl's Theorem we obtain that each V_i is finitely generated, that is, there exist $X_1^i, ..., X_{l_i}^i \in \mathbb{R}^S$ such that V_i is the cone generated by such securities, *i.e.,*

$$V_i = cone\left(\left\{X_1^i, ..., X_{l_i}^i\right\}\right).$$

This fact makes clear that even for the case where $\#S = 3$ we may have a financial market with an arbitrary large number of non-redundant assets. Of course, this possibility excludes the case of a financial market without bid-ask spreads by the well-known fact that the number of non-redundant securities in an incomplete market of securities without bid-ask spreads cannot be large than $\#S$.

Now, let us come back to the case given by a "pricing rule" generated by a ball as we have discussed before our Theorem 4 about finitely generated pricing rules.[24] Assuming, for simplicity, that the ball contains only strictly positive probabilities, we can invoke a result from Shneider and Wieacker (1981) showing that for an approximation small error of $\varepsilon > 0$ we need a polytope with many faces of order $(\varepsilon^{-1})^{(\#S-2)/2}$, which is exponential in $\#S$.[25] In another words, in order to approximate the ball capturing the potential set of risk-neutral probabilities through a sequence of errors with order n^{-1}, we should take a sequence of finitely generated pricing rules C_k represented by $\mathcal{Q}^k = co\left(\{P_i\}_{i=1}^k\right)$ such that k is of order $n^{(\#S-2)/2}$. This shows how complex should be a hypothetical underlying market related to a ball of risk-neutral probabilities.

More on the Completeness of Financial Markets

Let C be a finitely generated pricing rule as we have characterized in our Theorem 4. As soon as $F_C = L_C$ we known from ACF (2012) that the linear space of attainable claims or

[24] Note that if $\#S \leq 2$ then any ball is a polytope. So, let us consider the case $\#S > 2$.

[25] See also Cheang (2000) for an interesting discussion on the results about polytope and also for an alternative strategy of taking approximation through half-spaces.

else the marketed space of the underlying financial market induced from C, denoted by F, is defined without any ambiguity as $F = F_C$. So, in the case of no frictions (*i.e.,* $F_C = L_C$), the space of attainable claims is always F_C and indeed one has completeness if, and only if, F_C or equivalently L_C is equal to $\mathbb{R}^S$.

On the other hand, in general we have that $F_C \subset L_C$ and in this case we should to analyze the following possibilities in order to clarify the possibility of finding an underlying complete market of securities:

(a) When $L_C = \mathbb{R}^S$ we got that there is a unique possible market space F given by $\mathbb{R}^S$, and in this case we say that the market is efficiently complete.

(b) In the case where $L_C \neq \mathbb{R}^S = Span\,(L_C)$, we have obtained that there is a unique possible marketed space F, namely $F = Span\,(L_C)$.

(c) Finally, when $Span\,(L_C) \neq \mathbb{R}^S$, due to the next proposition, we get that the smallest possible marketed space F related to C is given by $Span\,(L_C)$.

Proposition 13 *For all finitely generated pricing rules C, given any underlying market $\mathcal{M}_C$ (*i.e.*, $C_{\mathcal{M}_C} = C$) the inclusion $Span\,(L_C) \subset F$ holds.*

We note that when $Span\,(L_C) \neq \mathbb{R}^S$, we can apply our Theorem 11 for obtaining the minimal market $\mathcal{M}_C = \left\{X_j, \left(q_j^A, q_j^B\right)\right\}_{j=0}^{m}$ such that $F = Span\,(L_C)$. Now, if we consider another market $\mathcal{M}' = \left\{X_j, \left(q_j^A, q_j^B\right)\right\}_{j=0}^{m+1}$ by taking $X_{m+1} \notin Span\,(L_C)$ with bid-ask prices q_{m+1}^A and q_{m+1}^B such that

$$q_{m+1}^B \leq -C\,(-X_{m+1}) \text{ and } q_{m+1}^A \geq C\,(X_{m+1})\,,$$

then C is also the super-replication price of $\mathcal{M}'$. Following the notion of effectiveness of a new security as proposed by Baccara, Battauz and Ortu (2006, BBO for short),[26] we get that X_{m+1} does not improve the super-hedging capability of investors. In this way, both markets $\mathcal{M}_C$ and $\mathcal{M}'$ have the same super-replication price and the marketed space related to $\mathcal{M}'$ is large than F, but these asset markets provides the same set of efficient securities. Furthermore,

[26] As proposed by BBO (2006), effectiveness of a new security occurs when this new traded security is long and short effective in the sense that it is optimal to take a long and a short position in the new security to super-replicate some future cashflow at the minimum cost. They show that no-arbitrage and effectiveness of X implies that the ask price of X must be smaller then the minimum cost incurred to super-replicate the payoff from a long position $C\,(X)$, and the its bid price must be larger than the maximum that can be borrowed against a liability not exceeding the one generated by a short position, which is given by $-C\,(-X)$. Also, they observe that effectiveness is a necessary condition for a position being efficient in the new market (BBO, 2006, p. 67).

note that from $\mathcal{M}_C$ we can take the similar strategy applied for obtaining $\mathcal{M}'$ and construct a market $\mathcal{M}'' = \left\{X_j, \left(q_j^A, q_j^B\right)\right\}_{j=0}^{m+n}$ with $X_j \notin Span\left(L_C\right)$ for all $j \in \{m+1, ..., n\}$, such that $Span\left(\{X_j\}_{0=1}^{m+n}\right) = \mathbb{R}^S$ and arbitrary bid-ask prices q_j^A and q_j^B such that

$$q_j^B \leq -C\left(-X_j\right) \text{ and } q_j^A \geq C\left(X_j\right).$$

Thus we get that the marketed space related to $\mathcal{M}''$ is complete but the set of efficient securities still given by L_C.

In order to illustrate this case where C is associated to L_C such that $Span\left(L_C\right) \neq \mathbb{R}^S$, consider the following class of pricing rules, for all $X \in \mathbb{R}^S$

$$C\left(X\right) = \left(1-\varepsilon\right) \sum_{s \in E_0} X\left(s\right) P\left(s\right) + \varepsilon P\left(E_0\right) \max_{s \in E_0} X\left(s\right) + P\left(E_0^c\right) \max_{s \in E_0^c} X\left(s\right),$$

where E_0 is a nonempty proper subset of S, $\varepsilon \in (0,1)$ and P is a strictly positive probability.[27]

This pricing rule is the super-replication price of the financial market $\mathcal{M}$ determinate by the frictionless bond S^* with price 1, the frictionless security given by the bet on E_0^c, denoted by $\left(E_0^c\right)^*$, with price $P\left(E_0^c\right) \in (0,1)$, and all Arrow-securities $\{s\}^*$, $s \in E_0$, where the pair of bid-ask prices for each $s \in E_0$ is given by

$$\left(q_s^A, q_s^B\right) = \left(\left(1-\varepsilon\right) P\left(s\right) + \varepsilon P\left(E_0\right), \left(1-\varepsilon\right) P\left(s\right)\right).$$

Hence, $\mathcal{M}$ is an incomplete market of securities (note that all Arrow-securities $\{s\}^*$ with $s \in E_0^c$ are not tradeable) with a frictionless bond such all tradeable Arrow-security has a positive bid-ask spread as specified above. Note also that the set of undominated securities satisfies

$$L_C = \bigcup_{s \in E_0} cone\left(\{S^*, S^*, \{s\}^*, -\{s\}^*\}\right) = Span\left(S^*, \{s\}^*;\ \text{ with } s \in E_0\right),$$

that is, $\mathcal{M}$ is minimal in the sense of generating the smallest marketed space compatible with C.

Now, if we consider the financial market $\mathcal{M}'$ by adding all Arrow-securities $\{s\}^*$ with $s \in E_0^c$ with respective pairs of bid-ask prices given by $\left(q_s^A, q_s^B\right) = \left(P\left(E_0^c\right), 0\right)$ then we get that $\mathcal{M}'$ is a complete market of securities but the super-replication capability of investor still as the same as in $\mathcal{M}$, and the set of efficient securities does not change after introducing such Arrow-securities with frictions as described above.

[27] It can showed also that this pricing rule can be rewritten as a Choquet pricing rule with respect the concave capacity v_C induced from C defined by, for all A

$$v_C\left(A\right) := C\left(A^*\right).$$

Complete markets with uniform bid-ask spreads

This section shows that for an interesting class of efficient complete market of securities, the pricing rule is the epsilon contaminated one. We find a condition, called uniform bid-ask spreads, equivalent to the epsilon-contaminated pricing rule asserting that the price of any security $X : S \to \mathbb{R}$ follows as

$$C(X) = (1-\varepsilon) E_P(X) + \varepsilon \max_{s \in S} X(s),$$

where, $P \in \Delta^+$ is a risk-neutral probability and $\varepsilon \in (0,1)$. This pricing rule states that the price of all securities are computed by taking always the same convex combination between its "pure price" $E_P(X)$ and the worst scenario payoff for the seller point of view. We note that, since $L_C = \mathbb{R}^S$, any underlying market $\mathcal{M}$ revealed by C must be an efficient complete market. Also, since $F_C = Span\{S^*\}$ any security that is not riskless has a positive bid-ask spread. Actually, this class of pricing rules has a strong property in terms of market feature.

Definition 14 *We say that $\mathcal{M} = \left\{X_j; \left(q_j^A, q_j^B\right)\right\}_{j=0}^m$ is a financial market of Arrow securities with a frictionless bond and uniform bid-ask spreads if:*

i) $X_0 = S^*$ with $q_0^A = q_0^B = 1$, $m = \#S$, and for all $j \in S$, $X_j = \{j\}^*$.

ii) For all $j \in S$, the bid-ask spread prices of $X_j = \{j\}^*$ is given by

$$q_j^A - q_j^B = \varepsilon.$$

iii) The trade strategy given by selling all Arrow securities and buying the bond generate a positive cost $\varepsilon > 0$, i.e.,

$$1 - \sum_{s=1}^{\#S} q_j^B = \varepsilon.$$

In the condition (i) we just impose that the bond is tradeable without bid-ask spreads and all Arrow securities are also available in the market. Condition (ii) says that every Arrow security has the same bid-ask spread which could be viewed as a very pessimistic valuation rule. In fact, in such financial markets, securities with lower bid price requires a relatively higher ask valuation than those securities with biggest bid price, *e.g.,* for a security market with uniform bid-ask given by 0,05€ , if for some Arrow security its bid price is 0,10€ then its ask price is 0,15€ (50% more) while for a Arrow security with bid price given by 0,20€ its ask price is 0,25€ (25% more). In fact, we note that for a general security X its bid-ask spread is given by $\varepsilon\left(\max_S X\left(s\right)-\min_S X\left(s\right)\right)$. The third condition gives a relation between the frictionless bond and the collection of Arrow securities, which provides a general rule for the feature of bid and ask prices of Arrow securities.

Our result that characterizes complete market of Arrow securities with uniform bid-ask spreads and a frictionless bond says that:

Theorem 15 *A complete market of Arrow securities $\mathcal{M}$ satisfies the uniform bid-ask spreads condition if, and only if, $\mathcal{M}$ is an underlying market of securities revealed by a pricing rule C that can be represented by a strictly positive probability P and a constant $\varepsilon \in (0,1)$, in the sense that for any security X*

$$C\left(X\right)=\left(1-\varepsilon\right)E_P\left(X\right)+\varepsilon\max_{s\in S}X\left(s\right).$$

It is worth noting that this pricing rule can be rewritten as the Choquet integral (see, for instance, CKL (1996))

$$C\left(X\right)=\int Xdv,$$

where v is the concave capacity given by

$$v\left(A\right)=\begin{cases}\left(1-\varepsilon\right)P\left(A\right)+\varepsilon, & A\neq\emptyset\\ 0, & A=\emptyset.\end{cases}$$

Actually, the risk-neutral capacity[28] v gives the ask-price of each bet A^* and its dual, defined by $\overline{v}\left(A\right)=1-v\left(A^c\right)$, describe the bid-price of each bet A^*.

It seems interest also to note the set of risk-neutral probabilities is given by

$$\mathcal{Q}=\left(1-\varepsilon\right)P+\varepsilon\Delta,$$

which can be also rewritten as the convex hull of all probabilities given by the following convex combination $\left(1-\varepsilon\right)P+\varepsilon\delta_{\{s\}}$, $s\in S$, that is

$$\mathcal{Q}=co\left(\left\{\left(1-\varepsilon\right)P+\varepsilon\delta_{\{s\}}\right\}_{s\in S}\right).$$

[28] ACF(2012) characterizes the class of all risk-neutral capacities related to pricing rules of incomplete markets without bid-ask spreads. In the case of frictionless complete markets the risk-neutral capacity becames a risk-neutral probability as in the fundamental theorem of asset pricing. See Cerreia-Vioglio, Maccheroni and Marinacci (2015) for an alternative characterization of Choquet pricing rules of complete markets based on the put-call parity. See also Castagnoli, Maccheroni, and Marinacci (2004).

Appendix

Part A: Finite Arbitrage-Free Securities Markets.

We consider a model in which there is a finite number of assets $X_j \in \mathbb{R}^S$, $0 \leq j \leq m$, with the possibility of bid-ask spreads, which is modeled by a couple of prices for each asset j given by $\left(q_j^A, q_j^B\right)$, where $q_j^A \geq q_j^B$. Also, we assume that $X_0 = S^* := (1, ..., 1)$ is the *riskless bond* with zero bid-ask spread, under the price normalization $q_0 = 1$ (*i.e.*, $q_0^A = q_0^B = 1$). Note that we are not allowing bid-ask at liquidation.

A portfolio of an agent is identified with a pair of vectors $\left(\theta^A, \theta^B\right) \in \mathbb{R}^{2(m+1)}$, where θ_j^A represents the number of units of asset j bought while θ_j^B represents the number of units of asset j sold by the agent.

We recall that an arbitrage opportunity is a portfolio strategy with no cost that yields a strictly positive profit in some states and exposes no loss risk.

Formally, a market $\mathcal{M} = \left\{X_j; \left(q_j^A, q_j^B\right)\right\}_{j=0}^m$ offer no-arbitrage opportunity if for any portfolio $\left(\theta^A, \theta^B\right) \in \mathbb{R}_+^{2(m+1)}$,

$$\begin{aligned}
\sum_{j=0}^m \left(\theta_j^A - \theta_j^B\right) X_j &> 0 \Rightarrow \sum_{j=0}^m \theta_j^A q_j^A - \sum_{j=0}^m \theta_j^B q_j^B > 0, \\
\sum_{j=0}^m \left(\theta_j^A - \theta_j^B\right) X_j &\geq 0 \Rightarrow \sum_{j=0}^m \theta_j^A q_j^A - \sum_{j=0}^m \theta_j^B q_j^B \geq 0.
\end{aligned}$$

An important result says that the market $\mathcal{M} = \left\{X_j, \left(q_j^A, q_j^B\right)\right\}_{j=0}^m$ offers no arbitrage opportunity if and only if there exists a (strictly positive) probability $P_0 \in \Delta^+$ such that $q_j^B \leq E_{P_0}(X_j) \leq q_j^A$, $0 \leq j \leq m$.

The set

$$\mathcal{Q}_{\mathcal{M}} = \{P \in \Delta^+ : q_j^B \leq E_P(X_j) \leq q_j^A, \forall j \in \{0, ..., m\}\},$$

is called the set of risk-neutral probabilities.

The pricing rule C generates by the market $\mathcal{M} = \left\{X_j, \left(q_j^A, q_j^B\right)\right\}_{j=0}^m$ is defined by the super-replication price given by, for all $X \in \mathbb{R}^S$

$$\begin{aligned}
C_{\mathcal{M}}(X) &= \inf \left\{ \sum_{j=0}^m \left(\theta_j^A q_j^A - \theta_j^B q_j^B\right) : \sum_{j=0}^m \left(\theta_j^A - \theta_j^B\right) X_j \geq X \right\} \\
&= \min \left\{ \sum_{j=0}^m \left(\theta_j^A q_j^A - \theta_j^B q_j^B\right) : \sum_{j=0}^m \left(\theta_j^A - \theta_j^B\right) X_j \geq X \right\}.[29]
\end{aligned}$$

It is worth noticing that for a securities market $\mathcal{M}$ offering no-arbitrage opportunity, the super-replication prices can be also represented by (Jouini and Kallal (1995))

$$C_{\mathcal{M}}(X) = \sup_{P \in \mathcal{Q}_{\mathcal{M}}} E_P(X), \text{ for all } X \in \mathbb{R}^S.$$

Hence, by taking the closure of the set of risk neutral probabilities $\mathcal{Q}^*_{\mathcal{M}} := \overline{\mathcal{Q}_{\mathcal{M}}}$, we have

$$C_{\mathcal{M}}(X) = \max_{P \in \mathcal{Q}^*_{\mathcal{M}}} E_P(X), \text{ for all } X \in \mathbb{R}^S.$$

Given a financial market $\mathcal{M} = \left\{X_j, \left(q_j^A, q_j^B\right)\right\}_{j=0}^m$, we say that $\mathcal{M}$ is complete if

$$\left\{\sum_{j=0}^m \left(\theta_j^A - \theta_j^B\right) X_j : \left(\theta^A, \theta^B\right) \in \mathbb{R}_+^{2(m+1)}\right\} = \mathbb{R}^S.$$

Part B: Proof of the Results in the Main Text

Proof of Theorem 4:

Assume that Q is a polytope, i.e., there exists a finite set $\{P_i\}_{i=1}^k$ s.t. $Q = co\left(\{P_i\}_{i=1}^k\right)$. First, we note that, for each $i \in \{1, ..., k\}$

$$V_i := \left\{X \in \mathbb{R}^S : C(X) = E_{P_i}(X)\right\}$$

is nonempty[30]. Also, V_i is a convex cone. Actually, we know that $C(\lambda X) = \lambda C(X)$ for all X and all $\lambda \geq 0$. Also, $X, Y \in V_i$

$$C(X+Y) \leq C(X) + C(Y) = E_{P_i}(X) + E_{P_i}(Y) = E_{P_i}(X+Y) \leq C(X+Y).$$

Hence, for all $\lambda \geq 0$ and all $X, Y \in V_i$

$$\lambda X + Y \in V_i.$$

Also, it is easy to see that each V_i is a polyhedral set, $\cup_i V_i = \mathbb{R}^S$ and $\alpha S^* \in V_i$ for all i.

[30] We assume that given a polytope $\mathcal{Q} = co\left(\{P_i\}_{i=1}^k\right)$ the number k is such that

$$k = \min\left\{n \in \mathbb{N} : \mathcal{Q} = co\left(\{P_i\}_{i=1}^n\right)\right\}$$

Now, assume that there exist a finite collection of polyhedral cones $\{W_l\}_{l=1}^{L}$ with the property that $\cup_l W_l = \mathbb{R}^S$ and such that for each l and for all $X, Y \in W_l$

$$C(X+Y) = C(X) + C(Y).$$

Note that for each $l \in \{1, ..., L\}$ the mapping $C : W_l \to \mathbb{R}$ can be represented by some $P_l \in \Delta$ in the sense that[31] $C(X) = E_{P_l}(X)$ for all $X \in W_l$.

We claim that $Q = co\left(\{P_l\}_{l=1}^{L}\right)$. Indeed, since[32]

$$\mathcal{Q} = \left\{P \in \Delta : E_P(X) \leq C(X), \forall X \in \mathbb{R}^S\right\},$$

it is easy to see that $co\left(\{P_l\}_{l=1}^{L}\right) \subset Q$. On the other hand, if there exists $P_0 \in Q \backslash co\left(\{P_l\}_{l=1}^{L}\right)$ then the Hahn-Banach separation theorem gives that there exists some $Y \in \mathbb{R}^S$ with

$$E_{P_0}(Y) > \max_{P \in co\left(\{P_l\}_{l=1}^{L}\right)} E_P(Y).$$

Note that since $\cup_l W_l = \mathbb{R}^S$, there exist l_0 s.t. $Y \in W_{l_0}$ and $C(Y) = E_{P_{l_0}}(Y)$, but

$$C(Y) \geq E_{P_0}(Y) > \max_{P \in co\left(\{P_l\}_{l=1}^{L}\right)} E_P(Y) \geq E_{P_{l_0}}(Y),$$

a contradiction. Hence $Q = co\left(\{P_l\}_{l=1}^{L}\right)$.

Proof of Theorem 5:

Assume that C is a super-replication price of an arbitrage-free financial market $\mathcal{M} = \left\{X_j, \left(q_j^A, q_j^B\right)\right\}_{j=0}^{m}$, that is, C can be write as

$$C(X) = \max_{P \in \mathcal{Q}_{\mathcal{M}}^*} E_P(X),$$

where

$$\mathcal{Q}_{\mathcal{M}}^* := \{P \in \Delta : q_j^B \leq E_P(X_j) \leq q_j^A, \forall j \in \{1, ..., m\}\},$$

contains at least one strictly positive probability.

[31] Recall that C is monotone and $S^* \in W_l$ for all l. So, we can apply the Lemma 1 in Kalai and Myerson (1977).

[32] Actually, Q is is the subdifferential of its support function at 0 (Schneider (1993), pp. 38), i.e., $\partial C(0) = \mathcal{Q}$. Recall that

$$\partial C(X) = \left\{v \in \mathbb{R}^S : C(Y) \geq C(X) + \Sigma v(s)(Y(s) - X(s)), \forall Y\right\}$$

Hence

$$\mathcal{Q}_{\mathcal{M}}^{*}=\left(\bigcap_{j=0}^{m}\{Y\in\mathbb{R}^{S}:\langle Y,X_{j}\rangle\leq q_{j}^{A}\}\right)\cap\left(\bigcap_{j=0}^{m}\{Y\in\mathbb{R}^{S}:\langle Y,-X_{j}\rangle\leq -q_{j}^{B}\}\right),$$

that is, $Q_{\mathcal{M}}^{*}$ is a bounded intersection of finitely many closed half-spaces and by Proposition 3.2.1 in Florenzano and Le Van (2001) the set $Q_{\mathcal{M}}^{*}$ is a polytope.

Now, assume that

$$C\left(X\right)=\max_{P\in\mathcal{Q}}E_{P}\left(X\right),$$

where $Q\subset\Delta$ is a polytope. Again, by the Proposition 3.2.1 in Florenzano and Le Van (2001) we have that there exist $m\in\mathbb{N}$ and $X_{j}\in\mathbb{R}^{S}$ and $b_{j}\in\mathbb{R}$ $(j=1,...,m)$ such that

$$Q=\{P\in\Delta:E_{P}\left(X_{j}\right)\leq b_{j}\text{ for all }j\in\{1,...,m\}\}.$$

Let

$$Q_{\mathcal{M}}^{*}=\left\{P\in\Delta:q_{j}^{A}\leq E_{P}\left(X_{j}\right)\leq q_{j}^{B}\text{ for all }j\in\{1,...,m\}\right\},$$

where

$$q_{j}^{A}:=\max_{P\in\mathcal{Q}}E_{P}\left(X_{j}\right)\text{ and }q_{j}^{B}:=\min_{P\in\mathcal{Q}}E_{P}\left(X_{j}\right).$$

It is straightforward to check that $Q_{\mathcal{M}}^{*}=Q$, which completes the proof, since indeed it is unmodified if $X_{0}=S^{*}$ and $q_{0}^{A}=q_{0}^{B}=1$ is added to the $X_{j}'s$ in case S^{*} does not initially belong to the $X_{j}'s$.

Before the proof of Theorem **6**, we provide an alternative and very useful characterization of L_{C}. This characterization says that a security X is undominated if, and only if, there exists a strictly positive linear pricing rule that prices it.

Theorem 16 *The set of undominated securities generated by a finitely generated pricing rule C satisfies*

$$L_{C}=\left\{X\in\mathbb{R}^{S}:\arg\max_{P\in\mathcal{Q}}E_{P}\left(X\right)\cap\Delta^{+}\neq\emptyset\right\}.$$

Proof of Theorem 16:

$(\supseteq)$: Assume that X is s.t. $\arg\max_{P\in\mathcal{Q}}E_{P}\left(X\right)\cap\Delta^{+}\neq\emptyset$, that is, there exists $P\in Q^{+}$ such that $C\left(X\right)=E_{P}\left(X\right)$. Given any $Y>X$ we get that

$$C\left(Y\right)=\max_{Q\in\mathcal{Q}}E_{Q}\left(X\right)\geq E_{P}\left(Y\right)>E_{P}\left(X\right)=C\left(X\right),$$

that is, $X \in L_C$.

$(\subseteq)$: Assume that $X \in \mathbb{R}^S$ satisfies $\arg\max_{P \in \mathcal{Q}} E_P(X) \cap \Delta^+ = \emptyset$. In this case, it is clear that $X \notin Span(S^*)$. We need to show that $X \notin L_C$. We denote by $\{P_1, ..., P_k\}$ the set of all extreme points and let w.l.o.g. $\{P_1, ..., P_l\} = \arg\max_{P \in \mathcal{Q}} E_P(X) \cap \{P_1, ..., P_k\}$.

Let $P \in co(\{P_1, ..., P_l\})$ such that $Supp[P] = \bigcup_{j=1}^{l} Supp[P_j]$. Since $E_P(X) = E_{P_1}(X) = ... = E_{P_l}(X) = C(X)$, we get from our hypothesis that $Supp[P] \subsetneq S$.

Let $A := Supp[P]^c$ and note that $A \neq \emptyset$. Given $\varepsilon > 0$, consider $Y_\varepsilon := X + \varepsilon A^*$. Hence, for all $j \in \{1, ..., l\}$ we have that $E_{P_j}(Y_\varepsilon) = E_{P_j}(X) = C(X)$. Also, note that for all $j \in \{l+1, ..., k\}$ we have that

$$C(X) > E_{P_j}(X)$$

and

$$E_{P_j}(Y_\varepsilon) = E_{P_j}(X) + \varepsilon P_j(A).$$

Hence,

$$\begin{aligned} C(Y_\varepsilon) &= \max\left\{\max_{1 \le j \le l} E_{P_j}(X), \max_{l+1 \le j \le k}\left[E_{P_j}(X) + \varepsilon P_j(A)\right]\right\} \\ &= \max\left\{C(X), \max_{l+1 \le j \le k}\left[\overbrace{E_{P_j}(X)}^{\overset{C(X)}{>}} + \varepsilon P_j(A)\right]\right\}. \end{aligned}$$

Moreover, we can choose $\varepsilon > 0$ sufficiently small such that $C(Y_\varepsilon) = C(X)$, but $Y_\varepsilon > X$ which implies that $X \notin L_C$.

Proof of 6:

First, recall that we have showed the following alternative characterization of L_C

$$L_C = \left\{X \in \mathbb{R}^S : \arg\max_{P \in \mathcal{Q}} E_P(X) \cap \Delta^+ \neq \emptyset\right\}.$$

So, an immediate consequence is that for all $X^0 \in \mathbb{R}^S$ we have that $Eff_{X^0}(C) \subset L_C$.

Now, consider $X \in L_C$. By the alternative characterization of L_C, we know that there exists $P \in \Delta^+$ such that $C(X) = E_P(X)$. Consider the order $\trianglerighteq$ over S given by

$$s \trianglerighteq s' \Leftrightarrow P(s) \geq P(s').$$

Take an enumeration of S given by $\{s_1, ..., s_n\}$ such that $s_1 \trianglerighteq s_2 \trianglerighteq ... \trianglerighteq s_n$. Now, consider X^0 given by the rule:

$$\begin{aligned} X^0(s_1) &:= 0, \text{ and for all } k \in \{2, ..., n\} \\ X^0(s_k) &:= X^0(s_{k-1}) + |X(s_k)| + X(s_{k-1}). \end{aligned}$$

We note that for all $k \in \{1, ..., n-1\}$ we have that

$$\begin{aligned} X(s_{k+1}) + |X(s_{k+1})| &\geq 0 \\ &\Rightarrow X(s_{k+1}) + X^0(s_k) + |X(s_{k+1})| + X(s_k) - \left(X(s_k) + X^0(s_k)\right) \geq 0 \\ &\Rightarrow X(s_{k+1}) + X^0(s_{k+1}) \geq X(s_k) + X^0(s_k). \end{aligned}$$

So, given $s, s' \in S$ such that $P(s) > P(s')$ then $s = s_k$ and $s' = s_{k+p}$ for some $k \geq 1$ and $p \geq 1$. Hence,

$$\begin{aligned} X(s_{k+p}) + X^0(s_{k+p}) &\geq ... \geq X(s_k) + X^0(s_k), \textit{ i.e.,} \\ X(s') + X^0(s') &\geq X(s) + X^0(s), \end{aligned}$$

which completes the proof.

Proof of Theorem 11:

Given $C : \mathbb{R}^S \to \mathbb{R}$, consider the sets F_C and L_C. If $F_C = L_C$ then by Araujo, Chateauneuf and Faro (2012), C is the pricing rule of an arbitrage-free financial market

$$\mathcal{M} = \left\{X_j, \left(q_j^A, q_j^B\right)\right\}_{j=0}^m,$$

where $q_j^A = q_j^B$ for all $j \in \{0, 1, ..., m\}$, and $Span\left(\{X_j\}_{j=0}^m\right) = F_C$.

Now, assume that $F_C \subsetneq L_C$. We know that there exists a finite set $\{P_1, ..., P_k\} \subset \Delta$ such that, by denoting $\mathcal{Q} = co(\{P_1, ..., P_k\})$,

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X).$$

Given $i \in \{1, ..., k\}$, we define the set of securities priced by P_i as

$$V_i := \left\{X \in \mathbb{R}^S : C(X) = E_{P_i}(X)\right\}.$$

Note that for all $i \in \{1, ..., k\}$, the set V_i is a convex cone. We also denote by

$$\begin{aligned} \{1, ..., k\}^+ &:= \left\{i \in \{1, ..., k\} : P_i \in \Delta^+\right\}, \text{ and} \\ \{1, ..., k\}^\partial &:= \left(\{1, ..., k\}^+\right)^c. \end{aligned}$$

Let us show that L_C is a finite union of convex cones. First, we note that[33]

$$L_C = \left(\bigcup_{i\in\{1,...,k\}^+} V_i \right) \cup \left(\bigcup_{i\in\{1,...,k\}^\partial} (V_i \cap L_C) \right).$$

Since each $V_i,\ i \in \{1,...,k\}^+$, is a convex cone we just need to show that for all $i_0 \in \{1,...,k\}^\partial$ the set $V_{i_0} \cap L_C$ is a finite union of convex cones.

Given $i_0 \in \{1,...,k\}^\partial$ consider the family of subsets

$$\mathcal{J}_{i_0} := \left\{ J \subset \{1,...,k\} :\ i_0 \in J \text{ and } co\left(\{P_j\}_{j\in J} \right) \cap \Delta^+ \neq \emptyset \right\}.$$

For all $J \in \mathcal{J}_{i_0}$ consider the set given by

$$V_{i_0}^J := \left\{ X \in V_{i_0} :\ E_{P_{i_0}}(X) = E_{P_j}(X) \text{ for all } j \in J \right\},$$

which allows to obtain that

$$V_{i_0} \cap L_C = \bigcup_{J\in\mathcal{J}_{i_0}} V_{i_0}^J.$$

Indeed, if $X \in V_{i_0} \cap L_C$ then $C(X) = E_{P_{i_0}}(X)$ and there exists $Q \in \Delta^+$ s.t. $C(X) = E_Q(X)$. We have that there exists $\alpha \in \Delta_+^{k-1}$ such that $Q = \sum_{i=1}^{k} \alpha_i P_i$. Now, consider

$$J_{i_0}^Q := \left\{ i \in \{1,...,k\} : \alpha_i > 0 \text{ and } E_{P_i}(X) = E_{P_{i_0}}(X) \right\}.$$

We note that if $J_{i_0}^Q = \emptyset$ then $E_{P_{i_0}}(X) > E_{P_i}(X)$ for all i s.t. $\alpha_i > 0$, which gives that

$$E_{P_{i_0}}(X) > \sum_{i=1}^{k} \alpha_i P_i = E_Q(X),$$

a contradiction, which allows to conclude that $J_{i_0}^Q \neq \emptyset$ and we obtain $X \in V_{i_0}^J$ where $J = J_{i_0}^Q$. On the other hand, if there exists $J \in \mathcal{J}_{i_0}$ s.t. $X \in V_{i_0}^J$ then since

$$L_C = \left\{ X \in \mathbb{R}^S : \arg\max_{P\in\mathcal{Q}} E_P(X) \neq \emptyset \right\},$$

we obtain that $X \in V_{i_0} \cap L_C$.

[33]Note that $\bigcup_{i\in\{1,...,k\}} V_i = \mathbb{R}^S$. Also, for all $i \in \{1,...,k\}^+$, $V_i \subset L_C$.

Hence, we have that there exists a collection of convex cones $\{W_\lambda\}_{\lambda\in\Lambda}$, $\#\Lambda < \infty$, such that

$$L_C = \bigcup_{\lambda\in\Lambda} W_\lambda.$$

Note that for all $\lambda \in \Lambda$ there exists $i_\lambda \in \{1,...,k\}^+$ such that $W_\lambda = V_{i_\lambda}$ or there exist $i_\lambda \in \{1,...,k\}^\partial$ and $J_\lambda \in \mathcal{J}_\lambda$ s such that $W_\lambda = V_{i_\lambda}^{J_\lambda}$.

From the definition of V_i and $V_{i_\lambda}^{J_\lambda}$ we have that all W_λ are polyhedral sets. Hence, by the Minkowski-Weyl's Theorem (See Rockafellar (1970), Theorem 19.1) we obtain that all W_λ are finitely generated, that is, there exist $X_1^\lambda, ..., X_{l_\lambda}^\lambda \in \mathbb{R}^S$ such that

$$W_\lambda = cone\left(\left\{X_1^\lambda, ..., X_{l_\lambda}^\lambda\right\}\right).$$

Summing up, this construction gives us the following finite collection of finite sets of securities $\left\{X_1^\lambda, ..., X_{l_\lambda}^\lambda\right\}_{\lambda\in\Lambda}$.

Consider the financial market $\mathcal{M}$ given by such family of securities, where the respective prices are given by[34]

$$\begin{aligned}\forall\ \lambda &\in \Lambda, \forall j \in \{1, ..., l_\lambda\}: \\ q_j^B &:= -C\left(-X_j^\lambda\right) = \min_{1\le i\le k} E_{P_i}\left(X_j^\lambda\right) \text{ and } q_j^A := C\left(X_j^\lambda\right) = \max_{1\le i\le k} E_{P_i}\left(X_j^\lambda\right).\end{aligned}$$

Note that since $S^* \in L_C$, we can assume that for some $\lambda \in \Lambda$ and some $j \in \{1, ..., l_\lambda\}$ one of the X_j is S^* and indeed that both the corresponding bid and ask prices are equal to 1. The (extended) set of risk neutral probabilities of this market is given by

$$\mathcal{Q}_{\mathcal{M}}^* = \left\{Q \in \Delta : \forall\lambda \in \Lambda \text{ and } \forall j \in \{1, ..., l_\lambda\},\ q_j^B \le E_Q\left(X_j^\lambda\right) \le q_j^A\right\}.$$

First, we note that $\mathcal{Q} \subset \mathcal{Q}_{\mathcal{M}}^*$. Assume that $P \in \mathcal{Q} = co\left(\{P_1, ..., P_k\}\right)$. Hence, there exists $\alpha = (\alpha_1, ..., \alpha_k) \in \Delta_+^{k-1}$ such that

$$P = \sum_{i=1}^k \alpha_i P_i.$$

Give $\lambda \in \Lambda$ and $j \in \{1, ..., l_\lambda\}$

$$\min_{i\in\{1,...,k\}} E_{P_i}\left(X_j^\lambda\right) \le \sum_{i=1}^k \alpha_i E_{P_i}\left(X_j^\lambda\right) \le \max_{i\in\{1,...,k\}} E_{P_i}\left(X_j^\lambda\right),$$

[34] Indeed, some X_j^λ may appear several times and therefore a suitable market would consider of all the different X_j^λ.

that is,

$$q_j^B \leq E_P\left(X_j^\lambda\right) \leq q_j^A,$$

which shows that $\mathcal{Q} \subset \mathcal{Q}_{\mathcal{M}}^*$. Hence, the market $\mathcal{M}$ is arbitrage-free. Let $C_{\mathcal{M}}$ be the super-replication price generated by the market $\mathcal{M}$. By the inclusion above we have that $C_{\mathcal{M}} \geq C$. We need to show the reverse inequality, *i.e.* that the equality $C_{\mathcal{M}} = C$ holds.

First, let us show that for all $X \in L_C, C_{\mathcal{M}}(X) = C(X)$. If $X \in L_C$ then there exists $\lambda \in \Lambda$ such that $X \in W_\lambda$. We note that $C : W_\lambda \to \mathbb{R}$ is such that there exists $i_\lambda \in \{1, ..., k\}$ such that $C(X) = E_{P_{i_\lambda}}(X)$ for all $X \in W_\lambda$. We note that that there exists $\phi = \left(\phi_1, ..., \phi_{l_\lambda}\right) \in \mathbb{R}_+^{l_\lambda}$ such that

$$X = \sum_{j=1}^{l_\lambda} \phi_j X_j^\lambda.$$

Hence,

$$\begin{aligned} C(X) &= E_{P_{i_\lambda}}(X) = \sum_{j=1}^{l_\lambda} \phi_j E_{P_{i_\lambda}}\left(X_j^\lambda\right) \\ &= \sum_{j=1}^{l_\lambda} \phi_j q_j^A \geq C_{\mathcal{M}}(X). \end{aligned}$$

Thus, for all $X \in L_C, C_{\mathcal{M}}(X) = C(X)$.

Now, assume that there exists $X \in \mathbb{R}^S$ such that $C_{\mathcal{M}}(X) > C(X)$. By our previous conclusion that $C_{\mathcal{M}}$ and C coincide over L_C we get that $X \notin L_C$. Now we claim that there exists $Y \in L_C$ such that $Y > X$ and $C(Y) = C(X)$. We note that it is enough to show that for any security X, setting

$$E_X := \left\{Y \in \mathbb{R}^S : Y > X \text{ and } C(Y) = C(X)\right\},$$

there exists $Y \in F_C \cap E_X$.

We note that since $X \notin L_C$ one gets $E_X \neq \emptyset$. Let us now prove that E_X is bounded from above, otherwise there would exist a sequence $\{Y_k\}_{k \geq 1}$, $Y_k \in E_X$, $\forall k \geq 1$ and $s_0 \in S$ such that $\lim_k Y_k(s_0) = +\infty$. Since there exists $P_0 \in \mathcal{Q}^+$ we get that

$$\begin{aligned} \lim_k C(Y_k) &\geq \lim_k E_{P_0}(Y_k) = \lim_k \sum_{s \in S} P_0(s) Y_k(s) \\ &\geq \sum_{s \neq s_0} P_0(s) X(s) + \lim_k P_0(s_0) Y_k(s_0) = \infty, \end{aligned}$$

but $C(Y_k) = C(X)$, $\forall k \geq 1$, which gives a contradiction.

Let us now show that E_X has a maximal element for the partial order $\geq$ on $\mathbb{R}^S$. Thanks to Zorn´s lemma we just need to prove that every chain $(Y_\lambda)_{\lambda \in \Phi}$ in E_X has an upper bound. Define Y by

$$Y(s) := \sup_{\lambda \in \Phi} Y_\lambda(s), \forall s \in S,$$

since E_X is bounded from above it implies that $Y \in \mathbb{R}^S$. It remains to check that $C(Y) = C(X)$. Let $\varepsilon > 0$ be given, and let $s_i \in S$, hence there is $\lambda_i \in \Phi$ such that $Y(s_i) \leq Y_{\lambda_i}(s_i) + \varepsilon$, since $(Y_\lambda)_{\lambda \in \Phi}$ is a chain there is $n \geq 1$ and $\widetilde{\lambda} \in \{\lambda_1, ..., \lambda_n\}$ such that $Y_{\widetilde{\lambda}} \leq Y \leq Y_{\widetilde{\lambda}} + \varepsilon S^*$, therefore $C\left(Y_{\widetilde{\lambda}}\right) \leq C(Y) \leq C\left(Y_{\widetilde{\lambda}}\right) + \varepsilon$, since $C\left(Y_{\widetilde{\lambda}}\right) = C(X)$ it turns out that $C(Y) = C(X)$.

Let now Y_0 be a maximal element of E_X, the proof will be completed if we show that $Y_0 \in L_C$, since indeed we already have $Y_0 > X$ and $C(Y_0) = C(X)$.

Let Y_1 be an arbitrary security such that $Y_1 > Y_0$ and let us show that $C(Y_1) > C(Y_0)$. Note that $Y_1 > X$ since $Y_1 > Y_0$ and $Y_0 > X$, therefore since Y_0 is a maximal element in E_X and $Y_1 > Y_0$ we have $C(Y_1) > C(X)$ hence $C(Y_1) > C(Y_0)$ since $C(Y_0) = C(X)$, which completes the proof of our claim.

Hence, there exists $Y \in L_C$ such that $Y > X$ and $C(Y) = C(X)$. So,

$$C_{\mathcal{M}}(Y) = C(Y) = C(X) < C_{\mathcal{M}}(X),$$

a contradiction with the fact that $C_{\mathcal{M}}$ is monotone and $Y > X$. Therefore, we can conclude that

$$C_{\mathcal{M}}(X) = C(X) \text{ for all } X \in \mathbb{R}^S.$$

Let us show finally that $Span\left(\left\{X_j^\lambda\right\}_{j \in \{1,...,l_\lambda\}}, \lambda \in \Lambda\right) = Span(L_C)$. Since for all $\lambda \in \Lambda$ and for all $j \in \{1, ..., l_\lambda\}$ we have that $X_j^\lambda \in L_C$ it is clearly that

$$Span\left(\left\{X_j^\lambda\right\}_{j \in \{1,...,l_\lambda\}}, \lambda \in \Lambda\right) \subset Span(L_C).$$

Let now $X \in Span(L_C)$, then

$$X = \sum_{w=1}^{w_0} \alpha_w X_w$$

for some $w_0 \in \mathbb{N}$, where $\alpha_w \in \mathbb{R}$ and $X_w \in L_C$ for all $w \in \{1, ..., w_0\}$. Hence, for each $w \in \{1, ..., w_0\}$ there exists some $\lambda^w \in \Lambda$ and some $\gamma^{\lambda^w} = \left(\gamma_1^{\lambda^w}, ..., \gamma_{l_{\lambda^w}}^{\lambda^w}\right) \in \mathbb{R}_+^{l_{\lambda^w}}$ such that

$$X_w = \sum_{l=1}^{l_{\lambda^w}} \gamma_l^{\lambda^w} X_l^{\lambda^w},$$

which gives that

$$X=\sum_{w=1}^{w_0}\alpha_w X_w=\sum_{w=1}^{w_0}\sum_{l=1}^{l_{\lambda^w}}\left(\alpha_w\gamma_l^{\lambda^w}\right)X_l^{\lambda^w}\in Span\left(\left\{X_j^{\lambda}\right\}_{j\in\{1,\ldots,l_\lambda\}},\lambda\in\Lambda\right).$$

Proof of Theorem 8:

First, we show that $(i)\Leftrightarrow(ii)$: Suppose that it is not true that $\mathcal{Q}\subset\Delta^+$. Hence, there exists $P\in\mathcal{Q}$ such that $Supp[P]\neq S$. Take $E:=Supp[P]$ which allows to obtain that $C(E^*)\geq P(E)=1$, that is, $E^*\notin L_C$ because $S^*>E^*$ and $C(S^*)=C(E^*)$, and we conclude that $L_C\neq\mathbb{R}^S$. Hence, if $L_C=\mathbb{R}^S$ then $\mathcal{Q}\subset\Delta^+$. For the converse, assume that $\mathcal{Q}\subset\Delta^+$. For all $X\in\mathbb{R}^S$, if $Y\in\mathbb{R}^S$ is such that $Y>X$ then $E_P(Y)>E_P(X)$ for all $P\in\mathcal{Q}$, and using the fact that $\mathcal{Q}$ is a nonempty and compact set we obtain that there exists $P^*\in\mathcal{Q}$ with $C(X)=E_{P^*}(X)$, and

$$C(Y)=\max_{P\in\mathcal{Q}}E_P(Y)\geq E_{P^*}(Y)>E_{P^*}(X)=C(X).$$

Now, we show that $(ii)\Leftrightarrow(iii)$: First, we note that, since $\mathcal{Q}\cap\cdot^+\neq\emptyset$, the fact that $\mathcal{Q}$ contains only full support probabilities is equivalent to the fact that all probabilities in $\mathcal{Q}$ are mutually absolutely continuous.

Assume that all measures in $\mathcal{Q}$ are mutually absolutely continuous, and let us to prove that C satisfies the KEM property.[35]

So let $X,Y,Z\in\mathbb{R}^S$, such that $C(X)=C(X\wedge Y)$. There exists $P^*\in\mathcal{Q}$ such that

$$C(X\wedge Y)=E_{P^*}(X\wedge Y)\leq E_{P^*}(X)\leq C(X)=C(X\wedge Y),$$

hence,

$$E_{P^*}(X)=E_{P^*}(X\wedge Y),$$

so, for all $s\in S=Supp[P^*]$, $Y(s)\geq X(s)$, which implies that

$$Y(s)\wedge Z(s)\geq X(s)\wedge Z(s)\ \text{ for all } s\in S,$$

and then we get the following state-wise equality

$$X(s)\wedge Y(s)\wedge Z(s)=X(s)\wedge Z(s)\ \text{ for all } s\in S,$$

[35]Note that KEM property can be rewriteen as, for all $X,Y,Z\in\mathbb{R}^S$,

$$C(X)=C(X\wedge Y)\Rightarrow C(X\wedge Z)=C(X\wedge Y\wedge Z).$$

allowing to conclude that

$$C(X \wedge Z) = C(X \wedge Y \wedge Z).$$

For obtaining the converse, assume that there exists $P_1, P_2 \in \mathcal{Q}$ and an event $E \subset S$ such that

$$P_1(E) = 0 < P_2(E).$$

Hence, taking $F = E^c$

$$P_1(F) = 1 > P_2(F).$$

Let $P_3 \in \arg\max_{P \in \mathcal{Q}} \{P(E)\}$, hence $P_3(E) \geq P_2(E) > 0$. We also note that $S^* \wedge F^* = F^*$ and

$$C(S^* \wedge F^*) = C(F^*) = 1 = C(S^*).$$

On the other hand,

$$F^* \wedge S^* \wedge E^* = 0 \text{ and } S^* \wedge E^* = E^*,$$

so that

$$C(S^* \wedge F^* \wedge E^*) = 0 < P_3(E) = C(S^* \wedge E^*),$$

that is,

$$C(S^*) = C(S^* \wedge F^*) \text{ and } C(S^* \wedge E^*) > C(S^* \wedge F^* \wedge E^*),$$

contradicting the KEM condition.

Proof of Proposition 13:

In order to show that $Span(L_C) \subset F$, we note that It is enough to show that $L_C \subset F$. Assume that $\mathcal{M} = \left\{X_j; \left(q_j^A, q_j^B\right)\right\}_{j=0}^m$ is a market, as discussed in our Appendix A, such that $C = C_{\mathcal{M}}$, that is, $F = Span\left(\{X_j\}_{j=0}^m\right)$ and

$$C(X) = \min\left\{\sum_{j=0}^m \left(\theta_j^A q_j^A - \theta_j^B q_j^B\right) : \sum_{j=0}^m \left(\theta_j^A - \theta_j^B\right) X_j \geq X\right\}.$$

Assume that $X \notin F$. Note that

$$C(X) = \sum_{j=0}^m \left(\overline{\theta}_j^A q_j^A - \overline{\theta}_j^B q_j^B\right),$$

for some portfolio $\left(\overline{\theta}_j^A, \overline{\theta}_j^B\right) \in \mathbb{R}_+^{2(m+1)}$ such that $\sum_{j=0}^m \left(\overline{\theta}_j^A - \overline{\theta}_j^B\right) X_j \geq X$. Since $X \notin F$ we obtain that $\sum_{j=0}^m \left(\overline{\theta}_j^A - \overline{\theta}_j^B\right) X_j > X$. Hence, by taking $Y := \sum_{j=0}^m \left(\overline{\theta}_j^A - \overline{\theta}_j^B\right) X_j$ we obtain that $Y > X$ and

$$C(Y) = \min\left\{\sum_{j=0}^{m}\left(\theta_j^A q_j^A - \theta_j^B q_j^B\right) : \sum_{j=0}^{m}\left(\theta_j^A - \theta_j^B\right) X_j \geq Y\right\}$$
$$\leq \sum_{j=0}^{m}\left(\overline{\theta}_j^A q_j^A - \overline{\theta}_j^B q_j^B\right) = C(X),$$

that is, by monotonicity, $C(X) = C(Y)$, which shows that $X \notin L_C$. In another words, if $X \in L_C$ then $X \in F$.

Proof of Theorem 15:

First, assume that $C(X) = (1-\varepsilon) E_P(X) + \varepsilon \max_{s \in S} X(s)$, we note that $F_C = span\{S^*\}$ and $L_C = \mathbb{R}^S$, in special, the bond is frictionless and all position X is efficient with positive bid-ask spread if X is not constant.

Now, consider the following financial market induced from C. Take $X_0 := S^*$, $m = \#S$ and $X_j := \{j\}^*$ where j denotes the state of nature $j \in S$. Also, $q_0^A = q_0^B = 1$ and for any $j \in \{1, ..., m\}$ set the ask price and the bid price, respectively,

$$q_j^A = C(\{j\}^*) \text{ and } q_j^B = -C(-\{j\}^*),$$

that is,

$$q_j^A = (1-\varepsilon) E_P(\{j\}^*) + \varepsilon \text{ and } q_j^B = (1-\varepsilon) E_P(\{j\}^*).$$

Hence, the extended set of risk-neutral probabilities is given by

$$\mathcal{Q}_{\mathcal{M}}^* = \{Q \in \Delta : (1-\varepsilon) P(s) \leq Q(s) \leq (1-\varepsilon) P(s) + \varepsilon \text{ for all } s \in S\}.$$

We also note that

$$\mathcal{Q} = (1-\varepsilon)\{P\} + \varepsilon\Delta = co\left(\left\{(1-\varepsilon)\{P\} + \varepsilon\delta_{\{s\}}\right\}_{s \in S}\right)$$

is a polytope and $\mathcal{Q} = \mathcal{Q}_{\mathcal{M}}^*$, which concludes this part of the proof.

Now, consider a efficient complete market of Arrow securities $\mathcal{M}$ that satisfies the uniform bid-ask spreads condition, that is, $X_0 = S^*$ with price $q^0 = 1$ and for all $j \in \{1, ..., \#S\}$, $X_j = \{j\}^*$, where there is a $\varepsilon \in (0,1)$ such that $q_j^A - q_j^B = \varepsilon$ and $1 - \sum_{s=1}^{\#S} q_s^B = \varepsilon$.

By defining for all $s \in S$,

$$P(s) := \frac{q_s^B}{1-\varepsilon} = \frac{q_s^A - \varepsilon}{1-\varepsilon},$$

and we note that

$$\sum_{s=1}^{\#S} P(s) = \sum_{s=1}^{\#S} \frac{\left(\sum_{s=1}^{\#S} q_s^B\right)}{1-\varepsilon} = 1.$$

Also, it is easy to show that $Q_{\mathcal{M}} = (1-\varepsilon)\{P\} + \varepsilon\Delta$, which entails that

$$C(X) = (1-\varepsilon) E_P(X) + \varepsilon \max_{s\in S} X(s).$$

References and Notes

1. Araujo, A., A. Chateauneuf, and J.H. Faro (2012): "Pricing rules and Arrow-Debreu ambiguous valuation," *Economic Theory,* 49, 1-35.

2. Arrow. K. (1953): *Le rôle des valeurs bousières pour la repartition la meillure des risques*, Colloq. Internat. Centre National de la Recherche Scientifique (Paris, 1952), Econometrie, 40, 41-47/ discussion, pp. 47-48, *CNRS* Paris, 1953. English translation: (1964) *Review of Economic Studies,* 31, 91-96.

3. Artzner, P., F. Delbaen, J.-M. Eber, D. Heath (1999): "Coherent Measures of Risk," *Mathematical Finance,* 9, 203-228.

4. Baccara, M. A. Battauz and F. Ortu (2006): "Effective securities in arbitrage-free markets with bid-ask spreads: a linear programming characterization," *Journal of Economic Dynamics and Control,* 30, 55-79.

5. Bensaid, B., J-P. Lesne, H. Pagés, and J. Scheinkman (1992): "Derivative asset pricing with transaction costs." *Mathematical Finance,* 2, 63-86.

6. Bewley, T. (2002): "Knightian decision theory: Part I," *Decisions in Economics and Finance,* 2, 79-110.

7. Boyle P., S. Feng, W. Tian, and T. Wang (2008): "Robust Stochastic Discount Factors, *The Review of Financial Studies,* 21, 1077-1122.

8. Castagnoli, E., F. Maccheroni and M. Marinacci (2002): "Insurance premia consistent with the market," *Insurance: Mathematics and Economics,* 31, 267-284.

9. Castagnoli, E., F. Maccheroni and M. Marinacci (2004): "Choquet insurance pricing: a caveat," *Mathematical Finance,* 14, 481-485.

10. Cerreia-Vioglio, S., F. Maccheroni and M. Marinacci (2015): "Put-Call parity and market frictions," *Journal of Economic Theory,* 157, 730-762.

11. Chateauneuf, A., R. Kast and A. Lapied (1996): "Choquet pricing for financial markets with frictions," *Mathematical Finance,* 6, 323–330.

12. Cheang, G. (2000): "A better approximation for balls," *Journal of Approximation Theory,* 104, 183-203;

13. Chew, S.H., and J.S. Sagi (2006): "Event exchangeability: Probabilistic shophistication without continuity or monotonicity," *Econometrica*, 74, 771-786.

14. Chew, S.H., and J.S. Sagi (2008): "Small words: Modeling attitudes towards sources of uncertainty," *Journal of Economic Theory,* 139, 1-24.

15. Dybvig, P. (1988a): "Distributional analysis of portfolio choice," *Journal of Business***,** 61, 369-393.

16. Dybvig, P. (1988b): "Inefficient dynamic portfolio strategies or how to throw a million dollars in the stock market," *Review of Financial Studies,* 1, 67-88.

17. Dybvig, P. H. and S. A. Ross (1986): "Tax clienteles and asset pricing," *Journal of Finance,* 41, 751-762.

18. Dybvig, P. H. and S. A. Ross (1982): "Portfolio efficient sets," *Econometrica,* 50, 1525-1546.

19. Epstein, L., and M. Marinacci (2007): "Mutual absolute continuity of multiple priors," *Journal of Economic Theory*, 137, 716-720.

20. Florenzano, M. and C. Le Van (2001): *Finite Dimensional Convexity and Optimization*, Springer.

21. Ghirardato, P. and M. Siniscalchi (2012): "Ambiguity in the small and in the large," *Econometrica* 80, 2827-2847.

22. Gilboa, I. and D. Schmeidler (1989): "Maxmin expected utility with a non-unique prior," *Journal of Mathematical Economics,* 18, 141-153.

23. Gilboa, I., F. Maccheroni, M. Marinacci, and D. Schmeidler (2010): "Objective and subjective rationality in a multiple prior model," *Econometrica,* 78, 755–770.

24. Hansen, L.P., and R. Jagannathan (1991): "Implications of security market data for models of dynamic economies.," *The Journal of Political Economy,* 99, 225-262.

25. Hansen, L.P., J. Heaton, J. Lee, and N. Roussanov (2007): "Intertemporal substitution and risk aversion," In *Handbook of Econometrics*, Vol. 6A. Chapter 61, 3967-4054.

26. Huber, P. J. (1981): *Robust Statistics*, Wiley Series in Probabilities and Mathematical Statistics, John Wiley & Sons.

27. Jouini, E. and H. Kallal (1995): "Martingales and arbitrage in securities markets with transactions costs," *Journal of Economic Theory,* 66, 178-197.

28. Jouini, E. (2000): "Price functionals with bid-ask spreads: An axiomatic approach," *Journal of Mathematical Economics,* 34, 547-558.

29. Jouini, E. and H. Kallal (2001): "Efficient trading strategies in the presence of market frictions," *The Review of Financial Studies,* 14, 343-369.

30. Kalai, E. and R. Myerson (1977): "Linear functionals of convex sets with applications to Economics, Game-Theory and Social Choice," Discussion paper No. 272, Northwestern University.

31. Kreps, D. (1979): "A representation theorem for 'preference for flexibility'," *Econometrica,* 47, 565-577.

32. Luttmer, E.G. (1996): "Asset pricing in economies with frictions," *Econometrica,* 64, 1439-1467.

33. Mukerji, S. and J.M. Tallon (2001): "Ambiguity aversion and incompleteness of financial markets," *Review of Economic Studies,* 68, 883-904.

34. Peleg, B. and M. E., Yaari (1975): "A price characterization of efficient random variables," *Econometrica* 43, 283-292.

35. Rinaldi, F. (2009): "Endogenous incompleteness of financial markets: The role of ambiguity and ambiguity aversion," *Journal of Mathematical Economics,* 45, 872-893.

36. Rockafellar, R. T., (1970): *Convex Analysis,* Princeton, University Press, Princeton, N.J.

37. Rigotti, L. and C. Shannon (2005): "Uncertainty and risk in financial markets," *Econometrica,* 73, 203-243.

38. Rigotti, L. and C. Shannon (2012): "Sharing risk and ambiguity," *Journal of Economic Theory,* 147, 2028-2039.

39. Schneider, R. (1993): "Convex Bodies: The Brunn-Minkowski Theory," *Encyclopedia of Mathematics and its Applications* 44. Cambridge Univ. Press.

40. Schneider, R. and J.A. Wieacker (1981): "Approximation of convex bodies by polytopes," *Bulletin of the London Mathematical Society,* 13, 149-156.

41. Werner, I. (1997): "Diversification and equilibrium in securities markets," *Journal of Economic Theory,* 75, 89-103.