



Ignorance and Competence in Choices Under Uncertainty

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Abstract

We propose a model of decision making that captures reluctance to bet when the decision maker (DM) perceives that she lacks adequate information or expertise about the underlying contingencies. On the other hand, the same DM can prefer to bet in situations where she feels specially knowledgeable or competent even if the underlying contingencies have vague likelihood. This separation is motivated by the Heath and Tversky's competence hypothesis as well as by the Fox and Tversky's comparative ignorance effect. Formally, we characterize preference relations \succsim over Anscombe-Aumann acts represented by

$$J(f) = \min_{p \in C} \int_A u(f) dp + \max_{p \in C} \int_{A^c} u(f) dp,$$

where u is an affine utility index on consequences, C is a nonempty, convex and (weak*) compact subset of probabilities measures, and A is a *referential chance event*. In this model there is a clear separation of ambiguity attitudes. The case $E \subset A$ captures possible familiar target events while the case $E \subset A^c$ might refer to the case of relative ignorance concerning related contingencies. This model captures a special case of event dependence of ambiguity attitudes in which the well known maxmin model is a special case. We also characterize the case where we have a Choquet Expected Utility representation. *Journal of Economic Literature Classification Number*: D81.

Key words: Ambiguity; Competence hypothesis; Comparative ignorance effect; Maxmin preferences; Choquet expected utility.

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1 Introduction

Motivated by the well-known Ellsberg paradox (1961), ambiguity became an important issue in decision theory that models sensitivity to the lack of precise probabilistic information. The most well known models capturing ambiguity sensitivity are given by preference relations with a non-additive functional representation, as in Schmeidler (1989)'s Choquet Expected Utility (CEU) and Gilboa and Schmeidler (1989)'s Maxmin Expected Utility (MEU) models. In this perspective, the classical additive case of Subjective Expected Utility (SEU) of Savage (1954) (or Anscombe and Aumann (1963)) imposes strong behavioral conditions on preferences, which includes independence, that implies in an insensitive or neutral attitudes towards ambiguity¹.

The widely discussed hypothesis that *people prefer to bet on known rather unknown probabilities* is the basis for the notion of ambiguity aversion (uncertainty aversion). For instance, this hypothesis is essential in the MEU model where a DM behaves *as if* having a set of probability measures that determines his *ex ante* valuation of any act by the corresponding worst expected utility². Although ambiguity aversion presents many interesting applications in economic problems³, the generality of this pattern of attitude towards ambiguity is questionable⁴. Heath and Tversky (1991) discussed another pattern of behavior where a DM might prefer to bet in a context that she considers themselves competent than in a context where she feels ignorant or uninformed. Here, the term competence is used in a broad sense that includes skill, knowledge or understanding. This ideas motivate Heath and Tversky (1991) to propose the "competence hypothesis" asserting that the DM's willingness to bet on an event depends not only on the estimated likelihood and the precision of that estimate, but also on her general knowledge or understanding of the relevant context. In the widely discussed Ellsberg urns, we have the situation of partial ignorance characterized the inability of improving the knowledge of the proportion of balls in the urn. Fox and Tversky (1995) extended the Heath and Tversky's analysis by asking what conditions produce ignorance aversion. The main idea in Fox and Tversky (1995) is that the DM's confidence betting on a target event is enhanced (diminished) when she contrasts her knowledge of the event with her inferior (superior) knowledge about another event, or when she compares himself with less (more) knowledgeable individuals. In this way, the "comparative ignorance hypothesis" of Fox and Tversky (1995) asserts that ambiguity aversion is driven by a comparison with more familiar sources of uncertainty or expert and it is diminished in the absence of such a comparison. Also, following again Heath and Tversky (1991), in many situations the DM's perception of his level of knowledge concerning a target event might be extremely positive

¹Ghirardato and Marinacci (2002) provided a complete characterization of a comparative notion in which the SEU model is the benchmark of ambiguity neutrality.

²Indeed, Cerreia-Vioglio et. al. (2011) provided a representation result for uncertainty averse preferences under a very weak notion of independence. For instance, special cases are given by Chateauneuf and Faro (2009) and Maccheroni, Marinacci and Rustichini (2006).

³See, for instance, Section 6 in Gilboa and Marinacci (2011).

⁴An interesting discussion on this topic is presented in Fox and See (2003).

and that case she also may prefer to bet on her vague assessment of familiar events rather than bet on chance events with matched probability. We aim to focus on the cases where an event A is a clear and unambiguous reference for the DM in terms of familiar or unfamiliar contingencies. Next, we illustrate situations in which the DM has a referential chance event that separates her possible patterns of behavior:

Example 1 *A bet is offered for a South American soccer commentator. It concerns quarterfinals in World Cup composed of four American teams and four European teams. He should bet on Cup champion. Since the commentator is an expert on American soccer, and he does not consider himself as a specialist on European soccer, he is optimistic in his success if an American team is chosen, and pessimistic in his success if an European team is chosen. Using the retrospect of the previous World Champions, the commentator considers that each continent has the same chance to win the World Cup, i.e., South America and Europe have 50% chance of winning. In fact, it sounds reasonable to suppose that it is not possible to assign a well defined probability for each team to be the champion. In this case, the results are ambiguous and they allow the commentator to behave differently depending on the event considered.*

Example 2 *A pulmonologist receives a patient with a undiagnosed disease. Before any exam, he will give some hypothesis in order to guide subsequent procedures. Despite his expertise in respiratory problems, the disease could be cardiac, for instance. In the preliminary diagnosis, the doctor needs to take into account whether the disease could be respiratory or not. Without accurate exams, the diagnosis involves uncertainty, because the disease could not be determined on probabilistic terms. Analyzing the frequency of patients with problems related to other specialities in his office, the pulmonologist considers a probability of 70% for a disease associated with the respiratory system. If the disease pointed out by the doctor is related to respiratory system, he will be optimistic about his prognostic. However, if the prognostic is related to cardiac system, then he will be pessimistic about his judgment. In this case, he will appoint a specialist in cardiac problems. The patient need not be informed about the process of medical decision. This paper studies a type of decision process and, in this case, this process could be only mental.*

Example 3 *A stock broker specialist in technology firms is hired by a stock brokerage. We assume that the stock broker prefers to handle technology firms assets than other firms assets. Here, uncertainty is related to the future prices of assets. Suppose that a bet is proposed for the stock broker in which he must point out the firm that will have the better performance in a group of ten assets of technologic and commodities firms. In general, assume that it is well known that 30% of the time the better performance is related to technologic firm. Nevertheless, it is unclear which is the likelihood associate to any specific asset to be the best.*

The common feature in these examples previously discussed is the existence

of a referential event, which can be viewed as an objective information that DMs obtain before making the decision.

In Example 3, the set of states of nature can be described by all possible performances of assets. Due to the behavior of the Stock Broker in face of the uncertainty associated with the assets future prices, he will split the states of nature in two groups of best performance: technology assets and commodity assets. He will be optimistic with the first group and pessimistic with the second one.

Formally, let S be the set of states of nature capturing the possible performances of all assets. Let A be the set of events in which the Stock Broker is pessimistic (management of commodity assets) and its complementary A^c the set of events in which the Stock Broker is optimistic (management of technological assets).

For the Stock Broker to take a comparison, it is reasonable to think that he has in mind a set of probability distributions in which he follows the forecast given by highest expected utility when he manages technology assets, otherwise he follows the forecast given by worst expected utility. In a general context, we propose a model in which, given an act $f : S \rightarrow X$, the DM behaves *as if* evaluating f according to the functional given by

$$J(f) = \min_{p \in C} \int_A u(f) dp + \max_{p \in C} \int_{A^c} u(f) dp,$$

where $u : X \rightarrow \mathbb{R}$ is an affine utility function and A is a referential event for the DM in terms of perception of her expertise or ignorance⁵.

In this representation, C is a set of probabilities that characterizes the DM's beliefs and the partition $\{A, A^c\}$ captures the separation of her attitudes toward uncertainty.

This paper is organized into four sections. After this introduction, the section 2 contains the notation and framework. The section 3 is devoted to the axioms, main theorem and the case of Choquet Expected Utility. The last section contains the Appendix with the proofs of our results.

2 Notation and Framework

Consider a set S of *states of nature*, endowed with a σ -algebra Σ of subsets or *events*, and a non-empty set X of *consequences*. We denote by \mathcal{F} the set of all (simple) *acts*: finite-valued functions $f : S \rightarrow X$ which are Σ -measurable⁶. Moreover, we denote by $B_0(S, \Sigma)$ the set of all real-valued Σ -measurable simple functions $a : S \rightarrow \mathbb{R}$. Given an event $A \in \Sigma$, the induced characteristic function is defined by the mapping $1_A : S \rightarrow \{0, 1\}$ with $1_A(s) = 1 \Leftrightarrow s \in A$. The

⁵Note that our model differs from the alpha-maxmin model as proposed by Ghirardato, Maccheroni and Marinacci (2004).

⁶Let \succsim_0 be a binary relation on X , we say that a function $f : S \rightarrow X$ is Σ -measurable if, for all $x \in X$, the sets $\{s \in S : f(s) \succsim_0 x\}$ and $\{s \in S : f(s) \succ_0 x\}$ belong to Σ .

norm in $B_0(S, \Sigma)$ is given by $\|a\|_\infty = \sup_{s \in S} |a(s)|$ (called sup norm) and we can define the space of all bounded and Σ -measurable functions by taken

$$B(S, \Sigma) := \overline{B_0(S, \Sigma)}^{\|\cdot\|_\infty},$$

i.e., $B(S, \Sigma)$ consists of all uniform limits of finite linear combinations of characteristic functions of sets in Σ (see Dunford and Schwartz, 1958, page 240). For any subset $K \subset \mathbb{R}$, we define $B_0(K) := \{a \in B_0(S, \Sigma) : a(s) \in K, \forall s \in S\}$, $B(K) := \overline{B_0(K)}^{\|\cdot\|_\infty}$, and $B^+ := B(\mathbb{R}_+)$.

A set-function $v : \Sigma \rightarrow [0, 1]$ is a capacity if: (i) $v(\emptyset) = 0$, $v(S) = 1$ and (ii) $\forall E, F \in \Sigma$ such that $E \subset F \Rightarrow v(E) \leq v(F)$. We say v is convex (concave) if for any $A, B \in \Sigma$,

$$v(A \cup B) + v(A \cap B) \geq (\leq) v(A) + v(B).$$

The conjugate of a capacity v is a capacity defined by $\bar{v}(A) := 1 - v(A^c)$, for all $A \in \Sigma$. It is easy to show that a capacity v is convex if, and only if, \bar{v} is concave. For a capacity v , we define:

$$\begin{aligned} \text{core}(v) & : = \{p \in \Delta : p(A) \geq v(A) \ \forall A \in \Sigma\}, \text{ and} \\ \text{acore}(v) & : = \{p \in \Delta : p(A) \leq v(A) \ \forall A \in \Sigma\}. \end{aligned}$$

It is also easy to see that $\text{core}(v) = \text{acore}(\bar{v})$.

A capacity p is a (finitely additive) probability when for any $E, F \in \Sigma$ such that $E \cap F = \emptyset$ we have that $p(E \cup F) = p(E) + p(F)$. We denote by $\Delta := \Delta(\Sigma)$ the set of all (finitely additive) probability measures $p : \Sigma \rightarrow [0, 1]$ endowed with the natural restriction of the well-known weak* topology $\sigma(ba, B)$.

Given a set of probabilities measures $C \subset \Delta$, we say that an event $A \in \Sigma$ is C -unambiguous if for all priors $p, q \in C$ it follows that $p(A) = q(A)$. The convex hull of a set C is defined by $\text{co}(C) := \bigcap \{D \subset \Delta : D \supset C \text{ and } D \text{ is convex}\}$.

Given a function $a \in B$, the Choquet integral of a with respect to v is given by

$$\int_S a dv := \int_{-\infty}^0 [v(\{a \geq \lambda\}) - 1] d\lambda + \int_0^{+\infty} v(\{a \geq \lambda\}) d\lambda,$$

where, $\{a \geq \lambda\} := \{s \in S : a(s) \geq \lambda\}$. For short, we denote, $\int a dv := \int_S a(s) v(ds)$. Of course, if v is a probability measure we obtain the usual notion of integration with additivity of integrals. Also, given a function $a \in B$, for any event $A \in \Sigma$, the integral of a over A is given by

$$\int_A a dv := \int a 1_A dv.$$

Clearly, note that $u(f) \in B_0(S, \Sigma)$ whenever $u : X \rightarrow \mathbb{R}$ and f belongs to \mathcal{F} , where the function $u(f) : S \rightarrow \mathbb{R}$ is the mapping defined by $u(f)(s) = u(f(s))$, for all $s \in S$.

Let x belong to X , define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. Hence, we can identify X with the set \mathcal{F}_c of constant acts in \mathcal{F} .

Additionally, we assume that X is a convex subset of a vector space. For instance, this is the case if X is the set of all finite-support lotteries on a set of prizes Z , as in the classic setting of Anscombe and Aumann (1963).

Using the linear structure of X we can define as usual for every $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$ the act:

$$\begin{aligned} \alpha f + (1 - \alpha)g & : S \rightarrow X \\ (\alpha f + (1 - \alpha)g)(s) & = \alpha f(s) + (1 - \alpha)g(s). \end{aligned}$$

Given $f, g \in \mathcal{F}$ and $A \in \Sigma$, fAg denote the act delivering the consequence $f(s)$ for $s \in A$ and $g(s)$ for $s \in A^c$. Also, we define the family of acts that are uncertain only over A by

$$\mathcal{F}_A = \{fAx : f \in \mathcal{F} \text{ and } x \in X\}.$$

The decision maker's preferences are given by a binary relation \succsim on \mathcal{F} , whose usual symmetric and asymmetric components are denoted by \sim and \succ . Finally, for any $f \in \mathcal{F}$, an element $x_f \in X$ is a *certainty equivalent* of f if $x_f \sim f$.

3 Axioms and Main Theorem

The class of preference that we propose here is characterized by the properties described in the axioms below.

3.1 Axioms

A1 Nontrivial Weak Order:

A1a (Completeness) For all f and g in \mathcal{F} : $f \succsim g$ or $g \succsim f$.

A1b (Transitivity) For all f, g and h in \mathcal{F} : If $f \succsim g$ and $g \succsim h$, then $f \succsim h$.

A1c (Nondegeneracy) There are f and g in \mathcal{F} , such that $f \succ g$.

A2 Certainty Independence: For all f, g in \mathcal{F} , $x \in X$ and for all α in $(0, 1)$:

$$f \succsim g \text{ implies } \alpha f + (1 - \alpha)x \succsim \alpha g + (1 - \alpha)x.$$

A3 Continuity: For all f, g and h in \mathcal{F} , the sets:

$$\{\alpha \in [0, 1] : \alpha f + (1 - \alpha)g \succsim h\} \text{ and } \{\alpha \in [0, 1] : h \succsim \alpha f + (1 - \alpha)g\}$$

closed in $[0, 1]$.

A4 Monotonicity: For all f and g in \mathcal{F} :

$$\text{If } f(s) \succ g(s) \text{ for all } s \in S, \text{ then } f \succ g.$$

A5 Event Dependence: There is a referential event $A \in \Sigma$, such that, for all f and g in \mathcal{F} , $x, y \in X$, and $\alpha \in (0, 1)$:

A5a (Uncertainty Aversion) If $fAx \sim gAx$, then $\alpha fAx + (1 - \alpha)gAx \succsim fAx$;

A5b (Uncertainty Seeking) If $fA^c x \sim gA^c x$, then $fA^c x \succsim \alpha fA^c x + (1 - \alpha)gA^c x$;

A5c (Separation) For all $\bar{x} \in X$, if $x \sim fA\bar{x}$ and $y \sim \bar{x}Af$, then

$$\frac{1}{2}x + \frac{1}{2}y \sim \frac{1}{2}f + \frac{1}{2}\bar{x}.$$

Nontrivial weak order, continuity and monotonicity are the same as the ones used in Anscombe and Aumann (1963). The Certainty Independence Axiom is the same as the one used in Gilboa and Schmeidler (1989). Recall that this axiom is more weaker than independence axiom fundamental for the Anscombe and Aumann (1963)'s representation. Moreover, this axioms allows the possibility of hedging, a notion impossible in the Independence axiom. In addition, the Ellsberg paradox violates the independence axiom because the preferences are reversed when is mixing with a nonconstant act.

Axiom *A5* is more general than the well know Uncertainty Aversion axiom fundamental for the Gilboa and Schmeidler (1989)'s representation. Nevertheless, this axiom considers the state space S divided in two complementary events, A and A^c , in which the DM has different attitudes towards uncertainty over \mathcal{F}_A and \mathcal{F}_{A^c} . In A and its sub-events, the DM is uncertainty averse, whereas for A^c and its sub-events the DM is uncertainty seeking (loving). Indeed, if $A = S$ or if for all $p \in C$ we have $p(A^c) = 0$ then Axiom *A5* implies the same behavioral pattern as the Uncertainty Aversion axiom of Schmeidler (1989).

Axiom *A5c* asserts that for all consequence $\bar{x} \in X$ and all act f , the induced acts $f_1 := fA\bar{x}$ and $f_2 := \bar{x}Af$ generate an average of the corresponding certainty equivalents which is indifferent to the mixture average of f and \bar{x} . Note that this property holds for the Subjective Expected Utility model for any pair of acts $f, g \in \mathcal{F}$. Actually, if \succsim is a SEU preferences, then

$$x \sim f \text{ and } y \sim g \implies \frac{1}{2}x + \frac{1}{2}y \sim \frac{1}{2}f + \frac{1}{2}g.$$

3.2 Main Theorem

Theorem 4 *Let \succsim be a binary relation on \mathcal{F} . Then, the following conditions are equivalent:*

- (1) *The preference relation \succsim satisfies the Axioms A1 - A5;*
- (2) *There exists an affine utility function $u : X \rightarrow \mathbb{R}$, and a nonempty, (weak*) compact, convex set C of finitely additive probability over Σ , and a C -unambiguous (referential) event $A \in \Sigma$ where the pair (u, C) represents \succsim in the sense that:*

$$f \succsim g \Leftrightarrow J(f) \geq J(g) \text{ for all } f, g \in \mathcal{F}$$

where,

$$J(f) = \min_{p \in C} \int_A u(f) dp + \max_{p \in C} \int_{A^c} u(f) dp.$$

Moreover, if another pair (u_1, C_1) also represents \succsim then there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u(\cdot) = \alpha u_1(\cdot) + \beta$ and $C = co(C_1)$.

This representation captures choices situations as those discussed in the Introduction. First, note that the event A is unambiguous with respect to C , that is, for all priors $p, p' \in C$, the agreement $p(A) = p'(A)$ holds. In this way, we call the event A as a *chance event*. In the soccer example, the commentator might behaves in accordance with this model. The chance event A is related to European teams and A^c is associated to South American teams, both events has 50% of occurrence. An important feature is that acts given by $f = xAy$, with $x, y \in X$, are "unambiguous" in the sense that the commentator evaluate such acts in a similar way of an expected utility agent. On the other hand, if an act f is uncertain over A or over A^c the commentator associates a pessimistic evaluation of this act over A and an optimistic evaluation of this act over A^c . For instance, the commentator might not be able to associates a well specified probability to the success of Brazil in the world championship. Since the commentator is uncertainty averse with respect to A , he associates the worst expected utility to bet on an European team. This happens because the commentator does not judge himself competent for this bet. On the other hand, with respect to the complementary events A^c and its sub-events, the commentator associates the best plausible expected utility for bet in a south American team.

In the same way, in the medical example, the reference event is the "general cause" of illness. Following the Introduction, the set A corresponds to cardiac illness and the set A^c corresponds to respiratory illness. Due to the frequency a patient appears with cardiac problems in his clinic, the specialist point out 70% of probability for the illness to be cardiac. Although he can infer about the nature of the illness through objective probabilities, the illness is uncertain. Thus, the doctor will give a higher weighting for his success in pointing out the illness when it is related with respiratory system and will underestimate his inference with a less weighting to success in case of a cardiac illness.

In the third example, the reference event is the company sector in which the stockbroker will manage the asset. The stockbroker should to decide who company his stock brokerage will invest in a set of companies comprised by commodities firms and technology firms. The stockbroker knows that 30% of the time the better achievement is related to technology companies, his speciality. Then, A represent the commodity companies and A^c the technology companies. Although he knows the probability that some technology firm has the better achievement, he does not know specifically which company will be the better. Given his specialty, he associates the better feasible expected utility for bet in a technology firm, represented in the functional by the component $\max_{p \in C} \int_{A^c} u(f) dp$, and the worst feasible expected utility for bet in a commodity firm, represented by the component $\min_{p \in C} \int_A u(f) dp$, because he considers himself competent for investments in technology firms and ignorant about invest in commodity firms.

3.3 The Case of *Choquet Expected Utility*

Our main result characterizes preferences relations \succsim with a multiple priors representation, where beliefs are modeled by a set of probability measures $C \subset \Delta$, and the reference event $A \in \Sigma$ is a chance event. In this Section we aim to characterize the special case given by $C = \text{core}(v)$, where v is a convex capacity. In this case, our representation J can be rewrite as

$$J(f) = \min_{p \in \text{core}(v)} \int_A u(f) dp + \max_{p \in \text{core}(v)} \int_{A^c} u(f) dp,$$

and by Schmeidler (1989),

$$J(f) = \int_A u(f) dv + \int_{A^c} u(f) d\bar{v}.$$

Also, we will see that this representation is also a special case of CEU model as proposed by Schmeidler.

Wakker (1990) proposed an elegant characterization of optimism and pessimism in the CEU model through comonotonicity, without imposing uncertainty aversion as proposed by Schmeidler (1990) and Chateauneuf (1991)⁷. In a similar way, we can impose conditions in order to obtain the CEU representation discussed above.

Let us provide the fundamental definitions for the next result:

Mixture-Independence: We say that the preference relation \succsim satisfies Mixture-Independence if for all $f, g, h \in \mathcal{F}$ and for all $\alpha \in (0, 1)$

$$f \succsim g \Leftrightarrow \alpha f + (1 - \alpha) h \succsim \alpha g + (1 - \alpha) h.$$

We say that the act $f, g \in \mathcal{F}$ are \succsim -comonotonic if there do not exist states $s, s' \in S$ such that

$$f(s) \succ f(s') \text{ and } g(s') \succ g(s).$$

Pessimism-Independence: We say that the preference relation \succsim satisfies Pessimism-Independence over $\mathcal{H} \subset \mathcal{F}$ if for all $f, g, h \in \mathcal{H}$, with g and h comonotonic, and for all $\alpha \in (0, 1)$

$$f \succsim g \Leftrightarrow \alpha f + (1 - \alpha) h \succsim \alpha g + (1 - \alpha) h.$$

Optimism-Independence: We say that the preference relation \succsim satisfies Optimism-Independence over $\mathcal{H} \subset \mathcal{F}$ if for all $f, g, h \in \mathcal{H}$, with f and h comonotonic, and for all $\alpha \in (0, 1)$

$$f \succsim g \Leftrightarrow \alpha f + (1 - \alpha) h \succsim \alpha g + (1 - \alpha) h.$$

We refer to Wakker (1990) for a discussion on the meaning of such notions of independence. Our characterizations follows as:

⁷See also Wakker (2001).

Theorem 5 Let \succsim be a binary relation on \mathcal{F} . Then, the following conditions are equivalent:

- (1) The preference relation \succsim satisfies the Axioms A1, A3, A4, A5c, Pessimism-Independence over \mathcal{F}_A , and Optimism-Independence over \mathcal{F}_{A^c} ;
- (2) There exists an affine utility function $u : X \rightarrow \mathbb{R}$, and a convex capacity v over Σ such that

$$f \succsim g \Leftrightarrow J(f) \geq J(g) \text{ for all } f, g \in \mathcal{F}$$

where,

$$J(f) = \int_A u(f) dv + \int_{A^c} u(f) d\bar{v}.$$

Moreover, u is unique up to a positive affine transformation, and v is uniquely determined. Furthermore, by considering for all $E \in \Sigma$,

$$\mu(E) := v(E \cap A) - v(E^c \cup A) + 1,$$

we have that

$$J(f) = \int_S u(f) d\mu.$$

4 Appendix

Proof of the Main Theorem

Part (1) \Rightarrow (2) is straightforward except Axiom A5. Part (2) \Rightarrow (1) will result from Lemma 1 to Lemma 4.

Axiom A5

Part A5a: Suppose $fAx \sim gAx$, then we have $J(fAx) = J(gAx)$. i.e.,

$$\begin{aligned} \min_{p \in \mathcal{C}} \int_A u(f) dp + u(x)p(A^c) &= \min_{p \in \mathcal{C}} \int_A u(g) dp + u(x)p(A^c) \\ \min_{p \in \mathcal{C}} \int_S u(f) \mathbf{1}_A dp &= \min_{p \in \mathcal{C}} \int_S u(g) \mathbf{1}_A dp. \end{aligned}$$

Since A is unambiguous,

$$\begin{aligned} J(\alpha fAx + (1 - \alpha)gAx) &= \min_{p \in \mathcal{C}} \int_A (\alpha u(fAx) + (1 - \alpha)u(gAx)) dp + u(x)p(A^c) \\ &= \min_{p \in \mathcal{C}} \int_A (\alpha u(f) + (1 - \alpha)u(g)) dp + u(x)p(A^c). \end{aligned}$$

Using $\min(a + b) \geq \min a + \min b$, we can write

$$\begin{aligned} &\min_{p \in \mathcal{C}} \int_S (\alpha u(f) \mathbf{1}_A + (1 - \alpha)u(g) \mathbf{1}_A) dp + u(x)p(A^c) \\ &\geq \alpha \min_{p \in \mathcal{C}} \int_A u(f) dp + (1 - \alpha) \min_{p \in \mathcal{C}} \int_A u(g) dp + u(x)p(A^c), \end{aligned}$$

so,

$$J(\alpha fAx + (1 - \alpha)gAx) \geq \alpha \min_{p \in C} \int_A u(f) dp + (1 - \alpha) \min_{p \in C} \int_A u(g) dp + u(x)p(A^c)$$

and,

$$\begin{aligned} \alpha \min_{p \in C} \int_A u(f) dp + (1 - \alpha) \min_{p \in C} \int_A u(g) dp + u(x)p(A^c) &= \min_{p \in C} \int_A u(f) dp + u(x)p(A^c) \\ &= J(fAx). \end{aligned}$$

i.e.,

$$\begin{aligned} J(\alpha fAx + (1 - \alpha)gAx) &\geq J(fAx) \\ \alpha fAx + (1 - \alpha)gAx &\succsim fAx. \end{aligned}$$

Part A5b: Suppose $fA^c x \sim gA^c x$, we have $J(fA^c x) = J(gA^c x)$. Thereby,

$$\begin{aligned} \max_{p \in C} \int_A u(f) dp + u(x)p(A) &= \max_{p \in C} \int_A u(g) dp + u(x)p(A) \\ \max_{p \in C} \int_S u(f) \mathbf{1}_A dp &= \max_{p \in C} \int_S u(g) \mathbf{1}_A dp. \end{aligned}$$

Since A is unambiguous,

$$\begin{aligned} J(\alpha fA^c x + (1 - \alpha)gA^c x) &= \max_{p \in C} \int_{A^c} (\alpha u(fA^c x) + (1 - \alpha)u(gA^c x)) dp + u(x)p(A) \\ &= \max_{p \in C} \int_{A^c} (\alpha u(f) + (1 - \alpha)u(g)) dp + u(x)p(A). \end{aligned}$$

Then, like the proof above, based on $\max(a + b) \leq \max a + \max b$, we can write,

$$\begin{aligned} &\max_{p \in C} \int_S (\alpha u(f) \mathbf{1}_{A^c} + (1 - \alpha)u(g) \mathbf{1}_{A^c}) dp + u(x)p(A) \\ &\leq \alpha \max_{p \in C} \int_{A^c} u(f) dp + (1 - \alpha) \min_{p \in C} \int_{A^c} u(g) dp + u(x)p(A), \end{aligned}$$

so,

$$J(\alpha fA^c x + (1 - \alpha)gA^c x) \leq \alpha \max_{p \in C} \int_{A^c} u(f) dp + (1 - \alpha) \max_{p \in C} \int_{A^c} u(g) dp + u(x)p(A)$$

and,

$$\begin{aligned} \alpha \max_{p \in C} \int_{A^c} u(f) dp + (1 - \alpha) \max_{p \in C} \int_{A^c} u(g) dp + u(x)p(A) &= \max_{p \in C} \int_{A^c} u(f) dp + u(x)p(A) \\ &= J(fA^c x). \end{aligned}$$

i.e.,

$$\begin{aligned} J(\alpha f A^c x + (1 - \alpha) g A^c x) &\leq J(f A^c x) \\ \alpha f A^c x + (1 - \alpha) g A^c x &\lesssim f A^c x. \end{aligned}$$

Part A5c: Since $0 \in u(X)$, there exists $\bar{x} \in X$ such that $u(\bar{x}) = 0$. Now, given $f \in \mathcal{F}$ and $x, y \in X$ such that $x \sim f A \bar{x}$ and $y \sim \bar{x} A f$. Because J represents \succsim , we obtain,

$$\begin{aligned} u(x) &= \min_{p \in C} \int_A u(f) dp + u(\bar{x}) p(A^c) \\ &= \min_{p \in C} \int_A u(f) dp \text{ and} \\ u(y) &= \max_{p \in C} \int_{A^c} u(f) dp + u(\bar{x}) p(A) \\ &= \max_{p \in C} \int_{A^c} u(f) dp. \end{aligned}$$

Thus,

$$\begin{aligned} J\left(\frac{1}{2}x + \frac{1}{2}y\right) &= \frac{1}{2}u(x) + \frac{1}{2}u(y) \\ &= \frac{1}{2} \min_{p \in C} \int_A u(f) dp + \frac{1}{2} \max_{p \in C} \int_{A^c} u(f) dp \\ &= \min_{p \in C} \int_A \left(\frac{1}{2}u(f A \bar{x}) + \frac{1}{2}u(\bar{x} A f)\right) dp + \max_{p \in C} \int_{A^c} \left(\frac{1}{2}u(f A \bar{x}) + \frac{1}{2}u(\bar{x} A f)\right) dp \\ &= J\left(\frac{1}{2}f A \bar{x} + \frac{1}{2}\bar{x} A f\right), \end{aligned}$$

then,

$$\frac{1}{2}x + \frac{1}{2}y \sim \frac{1}{2}f A \bar{x} + \frac{1}{2}\bar{x} A f.$$

Note that for all $s \in S$,

$$\frac{1}{2}f(s) A \bar{x} + \frac{1}{2}\bar{x} A f(s) = \begin{cases} \frac{1}{2}f(s) + \frac{1}{2}\bar{x}, & S \in A \\ \frac{1}{2}\bar{x} + \frac{1}{2}f(s), & S \in A^c \end{cases}$$

i.e.,

$$\frac{1}{2}f A \bar{x} + \frac{1}{2}\bar{x} A f = \frac{1}{2}f + \frac{1}{2}\bar{x}.$$

hence,

$$\frac{1}{2}x + \frac{1}{2}y \sim \frac{1}{2}f + \frac{1}{2}\bar{x}.$$

Part (2) implies (1):

Given $\succsim \subset \mathcal{F} \times \mathcal{F}$ satisfying A1 - A5, we need to find a representation $J : \mathcal{F} \rightarrow \mathbb{R}$ for \succsim , i.e.,

$$f \succsim g \Leftrightarrow J(f) \geq J(g)$$

By A1, A2 and A3, the preference $\succsim|_{X \times X}$ restricted to consequences, satisfies the well known conditions for the existence of a affine and nonconstant function

$$u : X \rightarrow \mathbb{R}$$

such that, for all $x, y \in X$,

$$x \succsim y \iff u(x) \geq u(y).$$

Moreover, we can assume that $0 \in \text{int}(u(X))$.

We note that for all $f \in \mathcal{F}$, there exists $x_f \in X$ such that $f \sim x_f$. Now, let $J : \mathcal{F} \rightarrow \mathbb{R}$ and for each $f \in \mathcal{F}$, $J(f) := u(x_f)$, where x_f is the certainty equivalent of f . Also, $B_0(S, \Sigma, u(X)) = \{u(f) : f \in \mathcal{F}\}$ ⁸.

The functional J defined over \mathcal{F} by $J(f) := u(x_f)$ allows us to define $I : B_0(u(X)) \rightarrow \mathbb{R}$, and if $a = u(f)$, then $I(a) = J(f)$.

Note that, I is well defined over $B_0(u(X))$: given a a such that $a = u(f)$ and $a = u(g)$, therefore $u(f(s)) = u(g(s))$, for all $s \in S$. By the monotonicity axiom, $f(s) \sim g(s)$ for all S implies $f \sim g$. Thus, $x_f \sim x_g$, which leads to $J(f) = J(g)$.

Lemma 6 *Let I be the functional over $B_0(u(X))$ induced from J representing \succsim . Then I can be extended to $B_0(\Sigma)$, and it satisfies:*

I.1. *I is normalized, i.e., for all $k \in u(X)$,*

$$I(k\mathbf{1}_s) = k.$$

I.2. *I is monotonic:*

$$a \geq b \implies I(a) \geq I(b).$$

I.3. *I is constant additive: For all $a \in B_0(u(X))$ and for all k such that $a + k\mathbf{1}_s \in B_0(u(X))$, $I(a + k) = I(a) + k$.*

I.4 *I is positively homogeneous, i.e., for all $k > 0$, $I(ka) = kI(a)$.*

I.5 *There exists $A \in \Sigma$ such that, for all $a, b \in B_0(u(X))$:*

$$\text{I.5a } I(aA0 + bA0) \geq I(aA0) + I(bA0);$$

$$\text{I.5b } I(0Aa + 0Ab) \leq I(0Aa) + I(0Ab);$$

$$\text{I.5c } I(a) = I(aA0) + I(0Aa).$$

Proof:

I.1 Let $k \in u(X)$, then exists some $x \in X$ such that $k = u(x)$ and $I(k\mathbf{1}_s) = I(u(x)\mathbf{1}_s) = u(x) = k$.

I.2 If $a = u(f)$ and $b = u(g) \in B_0(u(X))$ and $a \geq b$, then $u(f(s)) \geq u(g(s))$ for all $s \in S$. Thereby, by monotonicity we have $f \succsim g$, i.e., $J(f) \geq$

⁸See, for instance, Macheronni, Marinacci and Rusticinni (2006, p. 1478).

$J(g)$. This leads to the relation $I(a) = I(u(f)) = J(f) \geq J(g) = I(u(g)) = I(b)$.

I.3 By homogeneity, we can assume, without loss of generality, that $2a$ and $2k\mathbf{1}_s \in B_0(u(X))$. Let $\beta = I(2a) = 2I(a)$ and $u(f) = 2a$ for all $f \in \mathcal{F}$, taking $y, z \in X$ where $u(y) = \beta\mathbf{1}_s$ and $u(z) = 2k\mathbf{1}_s$. If $f \sim y$, by axiom *A5c*, we have,

$$\frac{1}{2}f + \frac{1}{2}z \sim \frac{1}{2}y + \frac{1}{2}z.$$

Therefore,

$$I(a + k\mathbf{1}_s) = I(\beta\mathbf{1}_s + k\mathbf{1}_s) = \frac{1}{2}\beta + k = I(a) + k$$

thus, I is additive constant.

I.4 Let $a = \alpha b$ where $a, b \in B_0(u(X))$ and $\alpha \in [0, 1]$. Let $g \in \mathcal{F}$ satisfying $u(g) = b$ and defines $f = \alpha g + (1 - \alpha)z$, with $z \in X$ and $u(z) = 0$. Then $u(f) = \alpha u(g) + (1 - \alpha)u(z) = \alpha b = a$, and $I(a) = J(f)$. By *A5c* axiom, $\alpha x_g + (1 - \alpha)z \sim \alpha g(1 - \alpha)z = f$, we have,

$$\begin{aligned} J(f) &= J(\alpha x_g + (1 - \alpha)z) \\ &= \alpha J(x_g) + (1 - \alpha)J(z) \\ &= \alpha J(x_g). \end{aligned}$$

Then, we can write,

$$I(\alpha b) = I(a) = J(f) = \alpha J(x_g) = \alpha I(b).$$

I.5a Let $a, b \in B_0(u(X))$. It is enough to show that

$$I\left(\frac{1}{2}aA0 + \frac{1}{2}bA0\right) \geq \frac{1}{2}I(aA0) + \frac{1}{2}I(bA0).$$

Given $f, g \in \mathcal{F}$ such that $u(fA\bar{x}) = aA0$ and $u(gA\bar{x}) = bA0$. If $I(aA0) = I(bA0)$, by uncertainty aversion over \mathcal{F}_A^9 , we obtain that

$$I\left(\frac{1}{2}aA0 + \frac{1}{2}bA0\right) \geq I(aA0) = \frac{1}{2}I(aA0) + \frac{1}{2}I(bA0).$$

Now, in the case of $I(aA0) > I(bA0)$, let $k = I(aA0) - I(bA0)$. We define $c = bA0 + k\mathbf{1}_s$, thus through the certainty independence axiom, we have $I(c) = I(bA0) + k = I(aA0)$. Applying the later axiom again and uncertainty axiom, we get

$$\begin{aligned} I\left(\frac{1}{2}aA0 + \frac{1}{2}bA0\right) + \frac{1}{2}k &= I\left(\frac{1}{2}aA0 + \frac{1}{2}c\right) \geq \frac{1}{2}I(aA0) + \frac{1}{2}I(c) = \frac{1}{2}I(bA0) + \frac{1}{2}k \\ &I\left(\frac{1}{2}aA0 + \frac{1}{2}bA0\right) \geq \frac{1}{2}I(aA0) + \frac{1}{2}I(bA0). \end{aligned}$$

⁹In this case, $\frac{1}{2}fA\bar{x} + \frac{1}{2}gA\bar{x} \succsim fA\bar{x}$.

I.5b Let $a, b \in B_0(u(X))$. It is enough to show that,

$$I\left(\frac{1}{2}0Aa + \frac{1}{2}0Ab\right) \geq \frac{1}{2}I(0Aa) + \frac{1}{2}I(0Ab)$$

Given $f, g \in \mathcal{F}$ such that $u(\bar{x}Af) = 0Aa$ and $u(\bar{x}Ag) = 0Ab$. If $I(0Aa) = I(0Ab)$, by uncertainty seeking over \mathcal{F}_{A^c} ¹⁰, we obtain

$$I\left(\frac{1}{2}0Aa + \frac{1}{2}bA0\right) \leq I(0Aa) = \frac{1}{2}I(0Aa) + \frac{1}{2}I(bA0).$$

Now, if $I(0Aa) > I(0Ab)$, let $k = I(0Aa) - I(0Ab)$. We define $c = 0Ab + k\mathbf{1}_s$, in which, by certainty independence, we have $I(c) = I(0Ab) + k = I(0Aa)$. Again, by certainty independence, and uncertainty seeking we get

$$\begin{aligned} I\left(\frac{1}{2}0Aa + \frac{1}{2}0Ab\right) + \frac{1}{2}k &= I\left(\frac{1}{2}0Aa + \frac{1}{2}c\right) \leq \frac{1}{2}I(0Aa) + \frac{1}{2}I(c) = \frac{1}{2}I(0Ab) + \frac{1}{2}k \\ &I\left(\frac{1}{2}0Aa + \frac{1}{2}0Ab\right) \leq \frac{1}{2}I(0Aa) + \frac{1}{2}I(0Ab). \end{aligned}$$

I.5c We need show that $I(a) = I(aA0) + I(0Aa)$.

Let $a \in B_0(u(X))$ where $a = u(f)$. Additionally, a can be written as,

$$a = u(\alpha f + (1 - \alpha)\bar{x}). \quad (1)$$

According to relation 1, we get

$$\begin{aligned} I(a) &= J(u(\alpha f + (1 - \alpha)\bar{x})) \\ &= J\left(u\left(\frac{1}{2}f + \frac{1}{2}\bar{x}\right)\right) \\ &= J\left(\frac{1}{2}x_{fA\bar{x}} + \frac{1}{2}x_{\bar{x}Af}\right) \\ &= \frac{1}{2}u(x_{fA\bar{x}}) + \frac{1}{2}u(x_{\bar{x}Af}) \\ &= \frac{1}{2}J(fA\bar{x}) + \frac{1}{2}J(\bar{x}Af) \\ &= \frac{1}{2}I(aA0) + \frac{1}{2}I(0Aa). \end{aligned}$$

Gilboa & Schmeidler (1989) showed that¹¹ there exists a unique and continuous extension of I to the whole $B(S, \Sigma)$ when I is monotonic, constant additive and positively homogeneous. Furthermore, this extension satisfies properties *I.1 - I.5*. as in our previous lemma. Recall that, given a set of probabilities $C \subset \Delta$, an event A is called a chance event if A is unambiguous w.r.t. C ($\forall p, q \in C, p(A) = q(A)$).

¹⁰In this case, $\frac{1}{2}\bar{x}Af + \frac{1}{2}\bar{x}Ag \preceq \bar{x}Af$.

¹¹For details about this extension and their proof, see their Lemma 3.4, p.147.

Proposition 7 *The functional $I : B(S, \Sigma) \rightarrow \mathbb{R}$ satisfies I.1 - I.5, if, and only if, there exists a set $C \subset \Delta$ closed (weak*), convex, nonempty and a chance event $A \in \Sigma$, such that,*

$$I(a) = \min_{p \in C} \int_A a \, dp + \max_{p \in C} \int_{A^c} a \, dp.$$

Proof:

Before to applying the classical results of MEU to our functional, we need to describe the dual of B_A and B_{A^c} .

For a given event A , note that the induced collection of events

$$\Sigma_A := \{E \in \Sigma : E \subseteq A \text{ or } E \cap A^c = A^c\}$$

is a σ -algebra.

Lemma 8 Σ_A is a σ -algebra.

Proof: We need to check the following properties:

1. $\emptyset, S \in \Sigma_A$;
2. If $E \in \Sigma_A$, then $E^c \in \Sigma_A$.
3. Given $(E_n)_{n \in \mathbb{N}}$ and $E_n \in \Sigma_A$, then $\bigcup_{n=1}^{\infty} E_n \in \Sigma_A$.

Clearly, $\emptyset, S \in \Sigma_A$ because $\emptyset \subseteq A$ and $S \cap A^c = A^c$. Now, given $E \in \Sigma_A$, clearly $E \subseteq A$ or $E \cap A^c = A^c$. In the case where $E \subseteq A$ then $E^c \cap A^c = A^c$. In fact, $E^c \cap A^c \subseteq A^c$ and if $s \in A^c$, then since $E \subseteq A$ we can say that $s \in E^c$, which leads to $s \in E^c \cap A^c$, i.e., $A^c \subseteq E^c \cap A^c$, or, $E^c \cap A^c = A^c$.

Now, if $E \cap A^c = A^c$, given $s \in E^c$, then $s \notin A^c$, because otherwise, we can write that $s \notin E \cap A^c$ and $s \in A^c$, which contradicts $E \cap A^c = A^c$.

Thus, for all $s \in E^c$, we have $s \in A$, i.e., $E^c \subseteq A$. This satisfies the second condition.

In order to prove the third condition, we only need to check if for all $n \in \mathbb{N}$, $E_n \subseteq A$, then $\bigcup_{n=1}^{\infty} E_n \subseteq A$.

Suppose there exists $n \in \mathbb{N}$ such that $E_n \not\subseteq A$, i.e., $E_n \cap A^c \neq \emptyset$.

We just need to show that,

$$\bigcup_{n=1}^{\infty} E_n \cap A^c = A^c.$$

In fact, the following relation can be written,

$$\bigcup_{n=1}^{\infty} E_n = \left(\bigcup_{n: E_n \subseteq A} E_n \right) \cup \left(\bigcup_{n: E_n \not\subseteq A} E_n \right)$$

then,

$$\begin{aligned} \left(\bigcup_{n=1}^{\infty} E_n \right) \cap A^c &= \left[\left(\bigcup_{n: E_n \subseteq A} E_n \right) \cup \left(\bigcup_{n: E_n \not\subseteq A} E_n \right) \right] \cap A^c \\ &= \left[\bigcup_{n: E_n \subseteq A} E_n \cap A^c \right] \cup \left[\bigcup_{n: E_n \not\subseteq A} E_n \cap A^c \right]. \end{aligned}$$

Since $\bigcup_{n: E_n \subseteq A} E_n \cap A^c = \emptyset$, we have,

$$\left(\bigcup_{n=1}^{\infty} E_n \right) \cap A^c = \left[\bigcup_{n: E_n \not\subseteq A} E_n \cap A^c \right] = \bigcup A^c = A^c.$$

This shows that the third condition is also satisfied. Therefore, Σ_A is a σ -algebra.

Consider the Banach spaces $B(S, \Sigma_A)$, $B(S, \Sigma_{A^c})$ and define

$$B_A(S, \Sigma) := \{a \in B(S, \Sigma) : a \text{ is constant in } A^c\}.$$

An very important result for our construction follows as:

Lemma 9 $B_A(S, \Sigma) = B(S, \Sigma_A)$.

Proof: First, we need to prove that any element in $B_A(S, \Sigma)$ also belongs to $B(S, \Sigma_A)$.

Let $b \in B_A(S, \Sigma)$, i.e., there exists $a \in B(S, \Sigma)$ and $k \in \mathbb{R}$ such that $b = aAk$.

Clearly, b is limited, because a is also bounded and $k \in \mathbb{R}$.

Now, we need to show that aAk is Σ_A -measurable, i.e., for all $r \in \mathbb{R}$, the set $a^{-1}((r, +\infty)) \in \Sigma_A$. By contradiction, if there is a real number r_0 such that $a^{-1}((r, +\infty)) \notin \Sigma_A$, then $a^{-1}((r, +\infty)) \not\subseteq A$ or $a^{-1}((r, +\infty)) \cap A^c \subsetneq A^c$. That is, $a^{-1}((r, +\infty)) \cap A^c \neq \emptyset$ and $a^{-1}((r, +\infty)) \cap A^c \subsetneq A^c$. So, $\emptyset \neq \{s \in A^c : a(s) > r_0\} \subsetneq A^c$. Then, there exists $\hat{s} \in A^c$ such that $a(\hat{s}) > r_0$, because $a^{-1}((r, +\infty)) \cap A^c \neq \emptyset$. On the other hand, since $a^{-1}((r, +\infty)) \cap A^c \subsetneq A^c$, there exists $\tilde{s} \in A^c$ such that $a(\tilde{s}) \leq r_0$. Thus, we conclude that a is not a constant function in A^c which is a contradiction. Hence, $B_A(S, \Sigma) \subset B(S, \Sigma_A)$.

For the converse, consider an arbitrary function $a \in B(S, \Sigma_A)$. We need to show that a is a constant function in A^c . Suppose the contrary, i.e., there exist $\tilde{r}, \hat{r} \in \mathbb{R}$ and $\tilde{s}, \hat{s} \in A^c$ such that,

$$a(\tilde{s}) = \tilde{r} > a(\hat{s}) = \hat{r}.$$

We have $\{s \in S : a(s) > \hat{r}\} \in \Sigma_A$, and $\hat{s} \notin \{s \in S : a(s) > \hat{r}\}$ which leads the following relation:

$$\{s \in S : a(s) > \hat{r}\} \cap A^c \neq A^c.$$

It is a contradiction, because,

$$\{s \in S : a(s) > \hat{r}\} \in \Sigma_A \text{ and } \Sigma.$$

Then, $a(A^c) = \{k\}$, with $k \in \mathbb{R}$. Also, $a = bAk$ for some $b \in B(S, \Sigma)$ and a is Σ -measurable because $a \in B(S, \Sigma_A) \subseteq B(S, \Sigma)$ and $A \in \Sigma$. This shows the sets $B_A(S, \Sigma)$ e $B(S, \Sigma_A)$ are the same.

Finally, it is obvious that $B_A(S, \Sigma) = B(S, \Sigma_A) \subseteq B(S, \Sigma)$. Thus, we have $I_1 : B(S, \Sigma_A) \rightarrow \mathbb{R}$ and $I_2 : B(S, \Sigma_{A^c}) \rightarrow \mathbb{R}$ with the properties already discussed. Furthermore,

$$\begin{aligned} B_A(S, \Sigma)^* &= B(S, \Sigma_A)^* = ba(S, \Sigma_A), \\ B_{A^c}(S, \Sigma)^* &= B(S, \Sigma_{A^c})^* = ba(S, \Sigma_{A^c}). \end{aligned}$$

Then $I_1 : B(S, \Sigma_A) \rightarrow \mathbb{R}$ satisfies the conditions of the main Lemma in Gilboa & Schmeidler (1989) and, therefore, there exists a nonempty, closed and convex set $C_1 \subset ba^1(S, \Sigma_A)$ such that for all $a \in B(S, \Sigma_A)$ ¹²

$$I_1(a) = \min_{p \in C_1} \int_S a \, dp.$$

Similarly, for $I_2 : B(S, \Sigma_{A^c}) \rightarrow \mathbb{R}$, we use the subadditive version in Gilboa and Schmeidler (1989), *i.e.*, there exists a nonempty, closed and convex set $C_2 \subset ba^1(S, \Sigma_{A^c})$ such that for all $b \in B(S, \Sigma_{A^c})$,

$$I_2(b) = \max_{p \in C_2} \int_S b \, dp.$$

We note that given a function $a \in B(S, \Sigma)$, we have that $aA0 \in B(S, \Sigma_A)$ and $0Aa \in B(S, \Sigma_{A^c})$. Therefore,

$$I_1(aA0) + I_2(0Aa) = \min_{p \in C_1} \int_A a \, dp + \max_{p \in C_2} \int_{A^c} a \, dp.$$

Now, by *I5c*,

$$I(a) = I(aA0) + I(0Aa) = I_1(aA0) + I_2(0Aa) = \min_{p \in C_1} \int_A a \, dp + \max_{p \in C_2} \int_{A^c} a \, dp.$$

So, we have proved that given for any function $a \in B(S, \Sigma)$, follows that

$$I(a) = \min_{p \in C_1} \int_A a \, dp + \max_{p \in C_2} \int_{A^c} a \, dp$$

where $C_1 \subseteq ba^1(S, \Sigma_A) =: \Delta_A$ and $C_2 \subseteq ba^1(S, \Sigma_{A^c}) =: \Delta_{A^c}$.

¹²Given a σ -algebra $\tilde{\Sigma}$ over S , $ba^1(S, \tilde{\Sigma})$ denotes the family of all probability measures over $\tilde{\Sigma}$.

Now, we need to find a nonempty, closed and convex set of probabilities $C \subset ba(S, \Sigma)$ such that,

$$I(a) = \min_{p \in C} \int_A a \, dp + \max_{p \in C} \int_{A^c} a \, dp.$$

Lemma 10 For all $A \in \Sigma$,

$$\Sigma_B \cap \Sigma_{B^c} = \{\emptyset, S, B, B^c\}.$$

Proof: The intersection of the set Σ_B with its complementary Σ_{B^c} , can be written as,

$$\Sigma_B \cap \Sigma_{B^c} = \{E \in \Sigma : [E \subseteq B \text{ or } E \cap B^c = B^c] \text{ and } [E \subseteq B^c \text{ or } E \cap B = B]\}.$$

We note that,

$$\Sigma_B \cap \Sigma_{B^c} = \{E \in \Sigma : [E \subseteq B \text{ and } E \subseteq B^c] \text{ or } [E \subseteq B \text{ and } E \cap B = B] \text{ or } [E \subseteq B^c \text{ and } E \cap B^c = B^c] \text{ or } [E \cap B^c = B^c \text{ and } E \cap B = B]\}.$$

Thus, if $E \in \Sigma$ is such that $[E \subseteq B \text{ and } E \subseteq B^c]$, clearly $E = \emptyset$. If $E \in \Sigma$ is such that $[E \subseteq B \text{ and } E \cap B = B]$, then $E = B$. If $E \in \Sigma$ is such that $[E \subseteq B^c \text{ and } E \cap B^c = B^c]$, $E = B^c$. And, finally, if $E \in \Sigma$ is such that $[E \cap B^c = B^c \text{ and } E \cap B = B]$, then $E = S$. This prove that $\Sigma_B \cap \Sigma_{B^c} = \{\emptyset, S, B, B^c\}$.

From the proposition above, we note that I_1 and I_2 over $B(S, \Sigma_A) \cap B(S, \Sigma_{A^c})$ are the same. Then, we have,

$$\begin{aligned} I_1(1A0) &= I_2(1A0) \\ I_1(0A1) &= I_2(0A1). \end{aligned}$$

Consequently,

$$\begin{aligned} \min_{p \in C_1} p(A) &= \max_{q \in C_2} q(A), \text{ and} \\ \min_{p \in C_1} p(A^c) &= \max_{q \in C_2} q(A^c), \text{ i.e.,} \\ \max_{p \in C_1} p(A) &= \min_{q \in C_2} q(A). \end{aligned}$$

Therefore, $\min_{p \in C_1} p(A) \leq \max_{p \in C_1} p(A) = \min_{q \in C_2} q(A) \leq \max_{q \in C_2} q(A) = \min_{p \in C_1} p(A)$, which shows the desirable equality.

Thus, A is unambiguous with respect to C_1 and unambiguous with respect to C_2 . Hence, there exists $r \in \mathbb{R}$ such that,

$$\begin{cases} \forall p, p' \in C_1, p(A) = p'(A) = r \text{ and} \\ \forall q, q' \in C_2, q(A) = q'(A) = r. \end{cases}$$

Now, defining the set

$$C = \{p \in \Delta : \exists p_1 \in C_1, \exists p_2 \in C_2 \text{ such that } p|_{\Sigma_A} = p_1 \text{ and } p|_{\Sigma_{A^c}} = p_2\}.$$

For $E \in \Sigma_A \cap \Sigma_{A^c}$, E is also unambiguous with respect to C .

Note that C is convex. In fact, for $p, q \in C$, there exists $p_1, q_1 \in C_1$ and exists $p_2, q_2 \in C_2$ such that $p|_{\Sigma_A} = p_1$, $p|_{\Sigma_{A^c}} = p_2$, $q|_{\Sigma_A} = q_1$, and $q|_{\Sigma_{A^c}} = q_2$.

Hence, $\alpha p + (1 - \alpha) q$, with $\alpha \in [0, 1]$, is such that $[\alpha p + (1 - \alpha) q]|_{\Sigma_A} = [\alpha p_1 + (1 - \alpha) q_1] \in C_1$ because C_1 is convex.

Similarly, $[\alpha p + (1 - \alpha) q]|_{\Sigma_{A^c}} = [\alpha p_2 + (1 - \alpha) q_2] \in C_2$ because C_2 is convex. Then C is convex. Also, $C \neq \emptyset$: Given an arbitrary pair of probabilities $p_1 \in C_1$ and $p_2 \in C_2$, and we can take p over Σ by where $p|_{\Sigma_A} = p_1$ and $p|_{\Sigma_{A^c}} = p_2$.

Remains to show that C is a closed set (weak*).

Given a net $\{p^\alpha\}$ such that $p^\alpha \in C$, for all α and $p^\alpha \xrightarrow{*} p$ is necessary shows that $p \in C$. We have $p^\alpha \xrightarrow{*} p \stackrel{(\text{def.})}{\iff} \forall a \in B(S, \Sigma)$,

$$\int a \, dp^\alpha \rightarrow \int a \, dp.$$

Since $p^\alpha \in C$, for all α , then given α , there exist $p_1^\alpha \in C_1$ and $p_2^\alpha \in C_2$ such that $p^\alpha|_{\Sigma_A} = p_1^\alpha$ and $p^\alpha|_{\Sigma_{A^c}} = p_2^\alpha$. Therefore, for all $a \in B(S, \Sigma_A) \subseteq B(S, \Sigma)$,

$$\int a \, dp^\alpha \rightarrow \int a \, dp.$$

So,

$$\int_A a \, dp^\alpha \rightarrow \int_A a \, dp$$

and,

$$\int_A a \, dp_1^\alpha \rightarrow \int_A a \, dp|_{\Sigma_A},$$

i.e., $p_1^\alpha \xrightarrow{*} p|_{\Sigma_A}$, since C_1 is closed, $p|_{\Sigma_A} \in C_1$.

Analogously, for all $a \in B(S, \Sigma_{A^c})$

$$\int a \, dp^\alpha \rightarrow \int a \, dp,$$

notably,

$$\int_A a \, dp^\alpha \rightarrow \int_A a \, dp.$$

In other words,

$$\int_A a \, dp^\alpha \rightarrow \int_A a \, dp|_{\Sigma_{A^c}},$$

i.e., $p_2^\alpha \xrightarrow{*} p|_{\Sigma_{A^c}}$. Since C_2 is closed $p|_{\Sigma_{A^c}} \in C_2$.

In short, $p^\alpha \xrightarrow{*} p$ with $p|_{\Sigma_A} \in C_1$ and $p|_{\Sigma_{A^c}} \in C_2$, i.e., $p \in C$, which shows that C is closed (weak*).

Now, note that for a function $a \in B_0(S, \Sigma_A)$, $a = \sum_{i=1}^N \alpha_i \mathbf{1}_{E_i} + k \mathbf{1}_{A^c}$ where $\alpha_i \in \mathbb{R}$ and $E_i \in \Sigma_A$ we have $(E_i)_{i=1}^N$ is a partition of A .

Then, for all $p_1 \in C_1$,

$$\int a \, dp_1 = \sum_{i=1}^N \alpha_i p_1(E_i) + k p_1(A^c)$$

and for all $p \in C$ there is a correspondent $p_1 \in C_1$ with $p|_{\Sigma_A} = p_1$. We can write,

$$\int a \, dp_1 = \sum_{i=1}^N \alpha_i p(E_i) + k p(A^c) = \int a \, dp.$$

Thus, $\min_{p_1 \in C_1} \int a \, dp_1 = \min_{p \in C} \int a \, dp$. If $a \notin B_0(S, \Sigma_A)$, we can take a sequence $(a_n)_{n \in \mathbb{N}} \subseteq B_0$ such that $a_n \xrightarrow{\|\cdot\|_\infty} a$.

Hence, given $p \in C$ there exists $p_1 \in C_1$ such that $p|_{\Sigma_A} = p_1$ and,

$$\int a \, dp = \lim_{n \rightarrow \infty} \int a_n \, dp = \lim_{n \rightarrow \infty} \int a_n \, dp_1 = \int a \, dp_1.$$

The procedure is similar for $a \in B(S, \Sigma_{A^c})$. Then we obtain that

$$I(a) = \min_{p \in C} \int_A a \, dp + \max_{p \in C} \int_{A^c} a \, dp,$$

which concludes the proof of our main result.

Proof of Theorem 5:

The proof of "if, and only if" follows from a combination of our Main Theorem with Lemma 7 and Corollary 8 (page 460) in Wakker (1990). Note that our condition A5c plays the same rule as in the proof of our Main Theorem.

For the equality

$$J(f) = \int_S u(f) \, d\mu,$$

note that it is enough to show that

$$I(a) = \int_S a \, d\mu,$$

for any function $a \geq 0$ because $a - \min_s a(s) \geq 0$ and constant additivity holds. Note that, from the definition of Choquet integral and that for all $E \in \Sigma$,

$$\mu(E) := v(E \cap A) - v(E^c \cup A) + 1,$$

we have

$$\begin{aligned}
\int_S a \, d\mu &= \int_0^\infty \mu(\{s \in S : a(s) \geq t\}) \, dt = \\
&= \int_0^\infty [v(\{s \in A : a(s) \geq t\}) - v(\{s \in S : a(s) < t\} \cup A) + 1] \, dt = \\
&= \int_0^\infty \{v(\{s \in A : a(s) \geq t\}) + [1 - v(\{s \in A^c : a(s) \geq t\}^c)]\} \, dt = (\Upsilon).
\end{aligned}$$

Also, note that both $\int_0^\infty v(\{s \in A : a(s) \geq t\}) \, dt$ and $\int_0^\infty \bar{v}(\{s \in A^c : a(s) \geq t\}) \, dt$ are finite, and then

$$\begin{aligned}
\int_S a \, d\mu &= (\Upsilon) = \int_0^\infty v(\{s \in A : a(s) \geq t\}) \, dt + \int_0^\infty \bar{v}(\{s \in A^c : a(s) \geq t\}) \, dt = \\
&= \int_A a \, dv + \int_{A^c} a \, d\bar{v}.
\end{aligned}$$

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