Dynamic Objective and Subjective Rationality

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Abstract

The objective and subjective rationality model characterizes decision makers (DMs) by two preference relations over uncertainty acts and provides a dual perspective of rationality. The first preference reflects choices that are rational in an objective sense and the second ones express choices labeled subjective rational. While an objective ranking means that the DM can convince others that she is right in making them, in a subjective choice the DM cannot be convinced that she is wrong in making them. Objective and subjective preferences are represented, respectively, by a Bewley's unanimity rule and a maxmin expected utility, both representations holding the same set of multiple priors.

We propose and axiomatize a dynamic Bayesian model for the objective and subjective rationality theory. The static model specifies some set of prior probabilities, which should be then updated in the light of new and relevant information. We provide two new axioms on the interplay of unconditional objective relations and conditional subjective preferences. Such axioms ensure that a conditional subjective relation is also a maxmin expected utility preference and the corresponding set of priors is derived from the full Bayesian updating, *i.e.,* it is generated by the prior-by-prior updating of all unconditional probabilities. Our main result thus provides a novel foundation for sequential consistent maxmin preferences as well as for the *full Bayesian updating*. Finally, we study the dynamics of objective preferences and its relations with our main result.

Keywords: Objective rationality, subjective rationality, multiple priors, sequential consistency, full Bayesian updating. *Journal of Economic Literature* Classification Number: D81.

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1 Introduction

The behavioral foundation of objective and subjective rationality under uncertainty proposed by Gilboa, Maccheroni, Marinacci and Schmeidler (2010) (henceforth, GMMS) shows how the Knightian decision theory of Bewley (2002) and the maxmin expected utility model of Gilboa and Schmeidler (1989) are complementary and can be combined in a rational preference formation perspective[1]. Objective rational choices mean that the DM can convince others that she is right in taking any of her declared preferred courses of actions. In another way, all (Bayesian) opinions that emerge from experts will agree with the decision made by the DM. An important fact is that, in many situations, some courses of actions cannot be solidly justified in the objective way and, at the same time, decisions have to be made. In such cases, subjective rational choices capture the cases where others cannot convince the DM that she is wrong in choosing the revealed desired course of actions. In special, in any subjective rational decision not every expert thinks that she is wrong in making it. Both notions of rationality aim to capture in conjunction the ability to convince of or to insist on one's opinion[2].

However, the GMMS approach does not address how the DM updates her objective and subjective preferences in response to new information about relevant events. Our main goal is to show that a *dynamic objective and subjective model of rationality* can be fully characterized in a Bayesian way. The static GMMS's model specifies some set of prior probabilities, which should be then updated in the light of new and relevant information about future contingencies. We provide two new axioms on the interplay of unconditional objective preferences, represented by Bewley's unanimity rules, and conditional subjective preferences. We assume that subjective preferences and the conceivable possible updating ones should be given by complete, transitive and continuous relations. The first axiom introduced in this paper is called *Intertemporal Consistency* (IC for short): For all acts f and g that deliver the same consequences on the complement of an event E, the IC axiom says that *if it is ex ante objective rational to choose f against g then it should be also subjective rational to choose f against g when conditioning on the event E*. The second axiom that we propose is called *Intertemporal Default to Certainty* (IDC for short): Given an act f which is potentially uncertain only over E and, otherwise, delivers a constant result x. The IDC axiom asserts that *if the DM cannot find objective compelling reasons to prefer the act f to the constant thing x in the unconditional context, then the DM will prefer to choose the constant outcome over the uncertain one after the event E has occurred.*

Our main result shows that Intertemporal Consistency and Intertemporal

[1] Motivated by Knight (1921), Bewley's model obtains a multiple priors model by relaxing only Completeness in the Anscombe and Aumann (1963)'s approach for expected utility. Also in the Anscombe and Aumann's context, but inspired by the well known Ellsberg (1961)'s paradox, Gilboa and Schmeidler's model proposed the maxmin expected utility theory by relaxing the Independence axiom (see, also, Schmeidler (1989) and Chateauneuf (1991)).

[2] An interesting discussion on rationality and uncertainty can be find in Postlewaite and Schmeidler (2012).

Default to Certainty imply that any conditional subjective relation is also a maxmin expected utility preference, and the corresponding set of conditional priors is determined by the full Bayesian update. Our main result thus provides a novel foundation for sequential consistent[3] maxmin preferences as well as for the prior-by-prior Bayesian updating rule.

The Static Model

Following the static GMMS model, the DM is characterized by two preferences relations denoted, respectively, by $\succsim^*$ and $\succsim^\#$. These preference relations are represented in a different way by the same utility index u and the same set of multiple priors C[4]. As we discussed before, the binary relation $\succsim^*$ reflects choices that are objectively rational: we say that it is objective rational to choose f against g if the DM can convince others that she is right in making it, or in another way, any expert will agree with this choice made by the DM. This interpretation is well captured by Bewley (2002)'s Knightian decision theory saying that $f \succsim^* g$ if and only if the expected utility of f is at least as high as that of g for any prior in the set C[5]. The second preference relation $\succsim^\#$ captures choices that are subjective rational. A necessary condition for being subjective rational to choose f rather than g is that experts cannot convince the DM that she is wrong in making it. In GMMS this idea is captured by a Gilboa and Schmeidler (1989)'s maxmin expected utility preference: $f \succsim^\# g$ if, and only if, the worst expected utility associated to f is at least as high as that of g when we consider whole priors in C. We note that this representation can be interpreted in the following way: The decision maker can find some opinion q in C generating a forecast about the expected utility of g which is dominated by all expected utility of f no matter the opinion that she takes from C[6]. In this way, if the DM cannot convince others about a decision involving the acts f and g and if she has to make a choice then f will be chosen against g if the worst opinion about g entails an expected utility not higher than all opinions about f. The fact that the same set of priors is considered in both representations is a consequence of two important axioms about the interplay between the objective and subjective relations. The first condition, Consistency, requires that a preference instance that is objective rational should be also subjective

[3] Sarin and Wakker (1998) proposed the notion of sequential consistency, which requires that if a DM has committed to a family of models then she uses the same family after conditioning to any (relevant) event.

[4] We might interpret the set of multiple priors as a set of opinion or subjective beliefs that emerges from a prior list of experts. See, for instance, Nascimento (2011).

[5] In another way, the worst opinion about the expected utility net of choosing f against g is non-negative, *i.e.,*

$$\min_{p \in C} \int (u(f) - u(g))\, dp \geq 0.$$

[6] Formally, $f \succsim^\# g$ iff there exists $q \in C$ such that for all $p \in C$

$$\int u(f)\, dp \geq \int u(g)\, dq.$$

rational. The second, Default to Certainty, asserts how subjective rationality completes between acts involving uncertainty and acts that do not through a "tie-breaking" rule that always favors constant against uncertainty.

The Dynamic Model

We propose and axiomatize a dynamic Bayesian model of objective and subjective rationality. For our main result (Theorem 1), first we set an unconditional objective preference $\succsim^*$ as discussed in the static model. An objective relevant event E is characterized by the fact that the DM can convince others that such event is not a miracle, *i.e.,* it has positive probability. Then, given any arbitrary objective relevant event E, we relate the unconditional objective preference $\succsim^*$ with the respective conditional subjective preference $\succsim_E^{\#}$, which is assumed to be only a complete, transitive and continuous binary relation. While $\succsim^*$ is given by a unanimity rule represented by a pair (u, C), we show that the axioms IC and IDC imply that $\succsim_E^{\#}$ is a maxmin expected utility preference with a representation involving the same utility index u, and a set of priors C^E generated just by taking all Bayesian updating of priors from the set C representing objective beliefs. This result contrasts with the fact that multiple prior models, in general, admit many different ways for updating[7]. Our main result thus provide a novel foundation for the *full Bayesian updating.*

We also analyze the dynamic objective preference *per se* and study the connections with our main result (Theorem 1). We argue that objective rationality should obey Objective Consequentialism (OC for short), Objective Ordinal Consistency (OOC for short), and Objective Dynamic Consistency (ODC for short). The former says that conditional on being informed of an event E, the DM can convince others that they should only care about contingencies in E. The axiom OOC guarantee that any conditional objective preference when restricted to consequences does not change. The axiom ODC says that the DM's power of persuasion about two acts differing only inside an event E does not depend on knowing or not that the complementary event of E is impossible. By assuming only that conditional objective preferences are given by continuous preorders, our axioms ensure that updated objective preferences are also given by Bewley's preferences (Proposition 2), which are represented by the same utility index and set of priors given by all Bayesian updating priors that come from the objective set of priors (Theorem 3). Our last result (Corollary 4) provides some equivalent ways that justify full Bayesian updating as the role of revising beliefs in the model of objective and subjective rationality.

2 Framework

Consider a set S of *states of nature (world)*, endowed with a σ-algebra Σ of subsets called *events*, and a non-empty set X of *consequences.* We denote by

[7] See, for instance, Pires (2002), Gilboa and Schmeidler (1995), and Ghirardato, Maccheroni and Marinacci (2007).

$\mathcal{F}$ the set of all the (simple) *acts*: finite-valued functions $f : S \to X$ which are Σ-measurable[8]. Moreover, we denote by $B_0(\Sigma)$ the set of all simple real-valued Σ-measurable functions $a : S \to \mathbb{R}$. The norm in $B_0(\Sigma)$ is given by $\|a\|_\infty = sup_{s\in S} |a(s)|$ (called *sup norm*) and will denote by $B(\Sigma)$ the supnorm closure of $B_0(\Sigma)$.

Given a mapping $u : X \to \mathbb{R}$, the function $u(f) : S \to \mathbb{R}$ is defined by $u(f)(s) = u(f(s)), \text{for all}\, s \in S$. We note that $u(f) \in B_0(S, \Sigma)$ whenever f belongs to $\mathcal{F}$.

Let x be a consequence in X, taking an abuse of notation we define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. Hence, we can identify X with the set of the constant acts in $\mathcal{F}$.

Additionally, we assume that the set of consequences X is a convex subset of a vector space. For instance, this is the case if X is the set of all simple lotteries on a set of *outcomes* Z. In fact, it is the classic setting of Anscombe and Aumann (1963) as re-stated by Fishburn (1970).

Using the linear structure of X we can define as usual for every $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$ the act:

$$\begin{aligned} \alpha f + (1-\alpha)g &: S \to X \\ (\alpha f + (1-\alpha)g)(s) &= \alpha f(s) + (1-\alpha)g(s). \end{aligned}$$

Also, given two acts $f, g \in \mathcal{F}$ and an event $E \in \Sigma$ we denote by fEg the act delivering the consequences $f(s)$ in E and $g(s)$ in $E^c := S\backslash E$ (the complement of E).

We denote by $\Delta := \Delta(\Sigma)$ the set of all (finitely additive) probability measures $p : \Sigma \to [0,1]$ endowed with the natural restriction of the well known weak* topology $\sigma(ba, B)$. Given an act $f \in \mathcal{F}$, a utility index u, and a probability measure $p \in \Delta$, the expected utility of f is given by $\int u(f)\,dp := \sum_{s\in S} u(f(s))\,p(\{s\})$. Moreover, we denote by Δ^σ the set of all countably additive probabilities in Δ. In particular, given $q \in \Delta^\sigma$, we denote by $\Delta^\sigma(q)$ the set of all probabilities in Δ^σ that are absolutely continuous w.r.t. q, *i.e.*, $\Delta^\sigma(q) = \{p \in \Delta^\sigma : p \ll q\}$, where $p \ll q$ means that $\forall A \in \Sigma$, if $q(A) = 0$ then $p(A) = 0$.

The decision maker's unconditional preferences are given by a binary relation $\succsim$ on $\mathcal{F}$, whose the usual symmetric and asymmetric components are denoted by $\sim$ and $\succ$.

For each $E \in \Sigma$, let $\succsim_E$ be the conditional preference of the decision maker given E. We say that an event E is *relevant* (w.r.t. $\succsim$) when if there exist two consequences such that $x \succ y$ then $xEy \succ y$ and $x \succ yEx$. For instance, assume that $\succsim$ is an expected utility preference on $\mathcal{F}$ represented by the utility functional $V(f) = \int u(f)\,dp$, then the fact $x \succ y$ implies $xEy \succ y$ gives us that $p(E) > 0$. For an relevant event E, we get a more general conclusion if we assume that the underlying preference $\succsim$ is a Bewley's preference or a maxmin

[8] Let $\succsim_0$ be a binary relation on X, we say that a function $f : S \to X$ is Σ-measurable if, for all $x \in X$, the sets $\{s \in S : f(s) \succsim_0 x\}$ and $\{s \in S : f(s) \succ_0 x\}$ belong to Σ.

expected utility preference: given the corresponding set of multiple priors C, $p(E) > 0$ for all $p \in C$.

Let p be a probability measure in Δ and $E \in \Sigma$ such that $p(E) > 0$. We denote by p^E its Bayesian updating or its conditional probability (w.r.t. E), which is defined by:

$$p^E(F) = \frac{p(E \cap F)}{p(E)}, \forall F \in \Sigma.$$

Also, the corresponding conditional expected utility of an act $f \in \mathcal{F}$ is given by

$$\int_S u(f)\, dp^E = \frac{1}{p(E)} \int_E u(f)\, dp.$$

We can extend the definition of Bayesian updating to sets of probability measures as follows. Let C be a set of probability measures and $E \in \Sigma$ such that $p(E) > 0$ for all $p \in C$. We denote by C^E the conditional probability set C given E, defined by the full Bayesian rule:

$$C^E = \{p^E | p \in C\}.$$

3 Model and Results

The objective and subjective rationality theory starts from the fundamental primitive given by two binary relations modeling the dual perspective of rationality discussed in the introduction. We denote by $\succsim^*$and $\succsim^\#$the objective and subjective preference relations, respectively. Assume that $E \in \Sigma$ is an objective relevant event[9]. After the DM is informed that the true state of nature lies in the event E, a natural question is what happens with both sides of rationality characterizing our DM. We denote the updating objective and subjective preferences, respectively, by $\succsim^*_E$ and $\succsim^\#_E$. The following diagram illustrates what we have.

$\succsim^*$	Unconditional preferences	$\succsim^\#$
$\downarrow$	The event E contains the true state	$\downarrow$
$\succsim^*_E$	Conditional preferences	$\succsim^\#_E$.

We begin with a discussion on the basic conditions imposed to all preference relations that we consider in this paper.

Basic Conditions: We say that a binary relation $\succsim$ on $\mathcal{F}$ satisfies *Basic Conditions* if:

1. $\succsim$ is a nontrivial, *i.e.,* there exist acts $f, g \in \mathcal{F}$ such that $f \succ g$.

2. $\succsim$ is reflexive, *i.e.,* for all acts $f \in \mathcal{F}$ we have that $f \succsim f$.

3. $\succsim$ is transitive, *i.e.,* given $f, g, h \in \mathcal{F}$, if $f \succsim g$ and $g \succsim h$ then $f \succsim h$.

[9] That is, E is relevant w.r.t. $\succsim^*$.

4. $\succsim$ is mixture-continuous, *i.e.,* given any $f, g, h \in \mathcal{F}$, the sets

$$\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim h\} \text{ and } \{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha) g\}$$

are closed in $[0,1]$.

Basic Conditions aim to capture minimal properties that objective and subjective relations should satisfy. We refer to GMMS (p. 759) for a detailed discussion on the properties characterizing Basic Conditions.

Unconditional Objective Preferences

Now, let us recall the axioms that are specific to unconditional objective rationality. We assume that the unconditional objective preference $\succsim^*$ satisfies Basic Conditions and also:

Monotonicity: For all $f, g \in \mathcal{F}$ if $f(s) \succsim g(s)\ \forall s \in S$, then $f \succsim g$.

C-Completeness: For all $x, y \in X$, $x \succsim^* y$ or $y \succsim^* x$.

Independence: For all $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1)$

$$f \succsim^* g \text{ iff } \alpha f + (1-\alpha) h \succsim^* \alpha g + (1-\alpha) h.$$

Monotonicity is a classic property also interpreted as state independence. The others conditions have been imposed by GMMS as natural properties concerning objective rationality. Actually, C-Completeness says that if the objective preference is incomplete then this does not come from any difficulties that the DM might have about determining her preference under certainty. Independence just follows its standard argument as discussed by GMMS (p. 757).

The Theorem 1 in GMMS (p. 760) presents the Ghirardato, Maccheroni and Marinacci (2004) characterization of unanimity rules: an objective preference $\succsim^*$ satisfies Basic Conditions, C-Completeness and Independence is equivalent to have that $\succsim^*$ is represented by an affine utility index $u : X \to \mathbb{R}$ and a closed and convex set of probabilities C such that[10], for all $f, g \in \mathcal{F}$

$$f \succsim^* g \Leftrightarrow \int u(f)\, dp \geq \int u(g)\, dp \text{ for all } p \in C.$$

The Dynamics of Subjective Rationality

We aim to analyze how *ex ante* objective rationality can support conditional subjective preferences. The main idea is that conditional subjective preferences should emerge from some regular properties between this class of preferences and the unconditional objective ones.

Consider an objective relevant event E, the subjective preference $\succsim^{\#}_E$ is assumed to satisfies Basic Conditions[11]. Also, we impose that $\succsim^{\#}_E$ satisfies:

Subjective Completeness: For all acts $f, g \in \mathcal{F}$, $f \succsim^{\#}_E g$ or $g \succsim^{\#}_E f$.

[10] The utility index u is unique up to affine transformations and C is unique.

[11] We note that when $E = S$ we get $\succsim^{\#}_S = \succsim^{\#}$.

The intuition is that in any informational context $E \subseteq S$, conditional or not, the subjective preference captures the fact that decisions have to be made. Even in the case where the DM cannot solidly justify her preference between two acts, the fact that $f \succsim_E^{\#} g$ or $g \succsim_E^{\#} f$ means that always the decision maker will have one choice and that others cannot convince her that she is wrong.

The first axiom relating unconditional objective and conditional subjective preferences is called:

Intertemporal Consistency: Given two acts $f, g \in \mathcal{F}$, if $fEg \succsim^* g$ then $f \succsim_E^{\#} g$.

Consider two acts h and g delivering the same consequence in each state of the complement of an objective relevant event E. This axiom establishes that if there is a unconditional objective proof that h is at least as good as g, then the same argument must works as a proof in favor of the same ranking in the conditional context. Clearly, if unconditional objective and subjective preferences coincide, then Intertemporal Consistency is exactly given by "one side" of the classical Dynamic Consistency axiom[12].

We note that in the unconditional case, $E = S$, we get (unconditional) Consistency[13], a fundamental property in the GMMS's model which appears also in Nehring (2001, 2009), titled compatibility.

Next axiom specifies another important property, which we call:

Intertemporal Default to Certainty: For any act $f \in \mathcal{F}$ and consequence $x \in X$,

$$\text{If not } fEx \succsim^* x \text{ then } x \succ_E^{\#} f.$$

This axiom also relates the *ex ante* objective rationality to any conceivable conditional subjective preference, but taking into account a restrictive class of comparisons. Give an objective relevant event E and an act h that is potentially uncertain only on E and delivers a constant x otherwise, in comparing the act $h = fEx$ and the constant x, the DM first checks whether there are compelling reasons to prefer fEx to x in the unconditional context. In the case where the DM can conclude that $fEx \succsim^* x$ then we should have $fEx \succsim_E^{\#} x$ from Intertemporal Consistency. If, however, no such reasons can be found in the unconditional context, after getting information about E the DM will opt for the constant act over the uncertain one[14]. We note that in the unconditional case, $E = S$, we get Default to Certainty, a fundamental property in the GMMS model[15]. Next, our main result (Theorem 1) shows that the extreme

[12]Note that in the GMMS model, both relations coincide if, and only if, we have a single prior representing objective rationality. Clearly, this is equivalent to the fact that the objective relation is complete, *i.e.,* it is a subjective expected utility preference.

[13]It says that for all $f, g \in \mathcal{F}$, if $f \succsim^* g$ then $f \succsim^{\#} g$.

[14]Clearly, since $\succsim_E^{\#}$ is complete, the Intertemporal Default to Certainty can be rewrite as *if* $f \succsim_E^{\#} x$ *then* $f \succsim^* x$. In this way, if conditional on E the DM cannot be convinced that she is wrong in choosing f against x then it has been available an objective proof that fEx is at least as good as x.

[15]Indeed, GMMS emphasizes much more a weaker axiom than Default to Certainty called "Caution": *If not* $f \succsim^* x$ *then* $x \succsim^{\#} f$. But, we note that in GMMS multiple prior model Default to Certainty and Caution are equivalent. Take a pair (u, C) representing a DM *a*

nature of Intertemporal Default to Certainty is reflected in the extremity of the maxmin rule characterizing any conditional subjective preference. Our main result characterizes conditional subjective preferences as follows:

Theorem 1 *Let $\succsim^*$ be a unconditional objective preference represented by a pair (u, C), E an arbitrary objective relevant event, and $\succsim^{\#}_E$ be a conditional subjective preference satisfying Basic Conditions and Completeness. The relations $\succsim^*$ and $\succsim^{\#}_E$ jointly satisfy Intertemporal Consistency and Intertemporal Default to Certainty if, and only if,*

$$f \succsim^{\#}_E g \;\Leftrightarrow \min_{q \in C^E} \int u(f)\, dq \geq \min_{q \in C^E} \int u(g)\, dq.$$

Our main result thus provides a novel foundation for the full Bayesian updating for maxmin expected utility preferences[16]. All conditional subjective preferences $\succsim^{\#}_E$, where E is objective relevant, follow the prior-by-prior updating rule applied to the set of priors representing the unconditional objective rationality[17]. Also, instead of postulating that conditional preferences preserve the structure of the *ex ante* ones, we obtain that Intertemporal Consistency and Intertemporal Default to Certainty yield conditional subjective preferences that also belong to the corresponding class of unconditional maxmin preferences representing subjective rationality as obtained by GMMS. This fact is in accordance with the sequential consistency proposed by Sarin and Wakker (1998)[18]. While the set C can be interpreted as representing *ex ante* "hard evidence", the set of conditional priors C^E derived via the full Bayesian updating rule represents which we can call *ex post* "hard evidence".

Dynamic Objective Preferences

The previous result, on the interplay of unconditional objective preference and conditional subjective ones, does not depend on the way in which the DM updates her objective preference. Next, we propose an axiomatic foundation for updating objective preferences in accordance with the intuition proposed by GMMS. Later, we will investigate the consistency of our results about revising objective preferences and the main result already discussed in this paper.

Given an objective relevant event $E \in \Sigma$, consider the conditional objective preference $\succsim^*_E$. We begging the exposition of the properties on the dynamics of objective preferences by:

la GMMS, then if not $f \succsim^* x$ then there exists $q \in C$ such that $\int u(f)\, dq < u(x)$. Hence, $u(x) > \int u(f)\, dq \geq \min_{p \in C} \int u(f)\, dp$, *i.e.*, $x \succ^{\#} f$. Also, by Faro (2013), the GMMS characterization still work under a weaker notion of Consistency.

[16] We note that when $E = S$ we get exactly the GMMS result, which can be viewed as a novel foundation of maxmin expected utility of Gilboa and Schmeidler (1989). See Theorem 4 (p. 762) in GMMS. See also Proposition 2 in Cerreia-Vioglio (2012).

[17] Epstein and Schneider (2003) provides an axiomatic foundation of *recursive* multiple priors obtaining also a prior-by-prior updating rule. This approach demands also a special structure for the sets of priors called 'rectangularity', which is not required in our result.

[18] See also, Gumen and Savochkin (2012) for a related concept, but more demanding, called dynamic stability.

Objective Consequentialism: For all acts $f, g \in \mathcal{F}$, $f \sim_E fEg$.

Objective Consequentialism says that, assuming that the true state of nature cannot lies outside of E, the DM can convince others that they should only care about contingencies in E. We will always assume that $\succsim^*$ satisfies consequentialism[19]. We also always assume that the DM's ordinal preference over consequences are identical over the family of objective relevant events[20], which can be presented as:

Objective Ordinal Consistency: Given an objective relevant event E and $x, y \in X$

$$x \succsim^* y \Leftrightarrow x \succsim^*_E y.$$

Now, we impose one of the most useful property in the study of dynamic preferences:

Objective Dynamic Consistency: For all acts $f, g \in \mathcal{F}$,

$$fEg \succsim^* g \;\Leftrightarrow\; f \succsim^*_E g.$$

This axiom expresses the fact that the DM's power of persuasion about any two acts, differing only inside an event E, does not depend on knowing or not that the complementary event E is impossible[21].

First, we get the following simple and powerful fact.

Proposition 2 *Assume that $\succsim^*$ satisfies Basic Conditions, C-Completeness, Independence, and that the pair $(\succsim^*, \succsim^*_E)$ jointly satisfy Objective Dynamic Consistency. Then $\succsim^*_E$ satisfies C-Completeness, Monotonicity, and Independence.*

Hence, Objective Dynamic Consistency gives us a way of updating objective preferences which guarantees that C-Completeness, Monotonicity, and Independence are properties of conditional objective preferences inherited from the unconditional ones[22]. Actually, in order to be able to convince others that $f \succsim^*_E g$ the DM can go back to the unconditional context and uses the fact that $fEg \succsim^* g$.

Clearly, $\succsim^*_E$ is also a unanimity rule, which by Objective Ordinal Consistency is also represented by the same utility index representing $\succsim^*$. Concerning the set of multiple priors representing $\succsim^*_E$, the following result obtains also a full Bayesian updating rule:

[19] Machina (1989) argues in favor of relaxing consequentialism in non-expected utility theory by its nonseparability nature. Hanany and Klibanoff (2007) studies the problem of updating multiple priors preferences without consequentialism.

[20] See, for instance, Ghirardato (2002).

[21] See Ghirardato, Maccheroni, and Marinacci (2008) for an approach in terms of *unambiguous preferences* as proposed in Ghirardato, Maccheroni, and Marinacci (2004) using a similar set of axioms. In GMMS (2008), they characterize full Bayesian updating for invariant biseparable preferences by looking at the induced unambiguous preference. See also a briefly discussion about Bewley preferences and updating in Epstein and Le Breton (1993).

[22] For an equivalence result between independence and dynamic consistency in the context of lotteries (under consequentialism and reduction of compound lotteries), see Karni and Schmeidler (1991). Also, Hammond (1988) showed that consequentialism is equivalent to Independence, but assuming implicitly that dynamic consistency and reduction of compound lotteries hold.

Theorem 3 *The objective preferences* $(\succsim^*, \succsim^*_E)$ *jointly satisfy Objective Dynamic Consistence if, and only if, the set of priors characterizing* $\succsim^*_E$ *is given by* C^E*, that is, for all* $f, g \in \mathcal{F}$

$$f \succsim^*_E g \Leftrightarrow \int u(f)\, dq \geq \int u(g)\, dq \text{ for all } q \in C^E.$$

The intuition is that after receiving information about a relevant event E, the DM take the *ex ante* justification based on a unanimity rule and updates her arguments taking into account all the previous ones in the light of new information[23]. In another way, the set C contemplates all *ex ante* opinions of experts while the updated set C^E is just the collection of all updated opinions of experts.

Summing up, given an objective relevant event E, the ranking $f \succsim^*_E g$ means that any expert agrees that the respect conditional expected utility of f is above the conditional expected utility of g. Moreover, the same ranking can also be justified by the fact that, *ex ante*, it was objective rational to choose the act $h := fEg$ against g. In terms of representation, this behavioral regularity is captured by taking the Bayesian updating of all unconditional opinions that emerge from experts.

Further on the Dynamics of Objective and Subjective Preferences

We have obtained that, under Objective Consistency and Objective Dynamic Consistency, the dynamic nature of objective rationality presents the same flavour of the standard Bayesian dynamic expected utility model, but with a generalized Bayesian rule. On the other hand, based in an intertemporal axiomatic relation between unconditional objective preferences and conditional subjective ones, our main Theorem 1 provides a novel foundation for the full Bayesian update. Next, we summarize our finds in a Corollary given some equivalent conditions:

Corollary 4 *Assume that the objective preference* $\succsim^*$ *has a Bewley's unanimity rule representation. The following conditions are equivalent:*

(i) For any objective relevant event E*, the pairing* $\left(\succsim^*, \succsim^\#_E\right)$ *jointly satisfies Intertemporal Consistency and Intertemporal Default to Certainty.*

(ii) For any objective relevant event E*, the pairing* $\left(\succsim^*_E, \succsim^\#_E\right)$ *jointly satisfies Consistency and Default to Certainty, and the objective preferences* $(\succsim^*, \succsim^*_E)$ *jointly satisfies Objective Dynamic Consistency.*

(iii) For any objective relevant event E*, the pairing* $\left(\succsim^*_E, \succsim^\#_E\right)$ *jointly satisfies Consistency and Default to Certainty, and the subjective preferences* $\left(\succsim^\#, \succsim^\#_E\right)$ *jointly satisfies Constant Dynamic Consistency,* i.e.*, for all act* $f \in \mathcal{F}$ *and consequence* $x \in X$*,*

$$fEx \succsim^\# x \text{ if, and only if, } f \succsim^\#_E x.$$

[23] In the Appendix, we provide a more general result that can be applied to others models. See Lemma 5 and Remark 2.

Condition (*i*) is exactly the behavioral properties present in our Theorem 1 on the interplay of unconditional objective preferences and conditional subjective relations with a maxmin expected utility representation satisfying the full Bayesian updating rule. In condition (*ii*), we impose the properties about objective preferences as in our Theorem 3, and also that atemporal Consistency and Default to Certainty hold both for the *ex ante* and *ex post* objective and subjective preferences. The condition (*iii*) also imposes that the atemporal interplay between objective and subjective preferences of GMMS holds *ex ante* and *ex post*, but differently from condition (*ii*) it imposes a dynamic regularity between unconditional and conditional subjective preferences given by the Constant Dynamic Consistency axiom[24]. Hence, by (*i*) and (*ii*), we can also view our main Theorem 1 as a behavioral justification for *ex ante* and *ex post* Consistency and Default to Certainty, at the same time that our DM updates her objective preference following Dynamic Consistency. On the other hand, by (*i*) and (*iii*), a similar reasoning says that our Theorem 1 guarantee the *ex ante* and *ex post* Consistency and Default to Certainty *vis a vis* the fact that subjective preferences are updating following Constant Dynamic Consistency. Furthermore, Corollary 4 shows the robustness of Theorem 1 by providing different ways of obtaining full Bayesian updating as the role of revising beliefs under the perspective of objective and subjective rationality theory.

4 Conclusion

We propose and axiomatize a dynamic Bayesian model for objective and subjective rationality theory of GMMS. To the best of our knowledge, this is the first attempt to provide a theory of updating beliefs in a model dealing with a pair of preference relations. Actually, Ghirardato, Maccheroni, and Marinacci (GMM; 2004, 2008) proposed a model with a primitive given by a *single* binary relation that may exhibit nonneutrality to ambiguity, which induces a "unambiguous preference". By requiring essentially Dynamic Consistency, the unambiguous preference is then updated in accordance with the full Bayesian updating. This result can be related, in terms of mathematical structure, to our result on objective preferences (Theorem 3), but the emphasis and motivation are quite different. Also, there is no direct relation between our Theorem 1 and the GMM work[25].

In our Theorem 1, by assuming that unconditional objective preferences

[24] For further discussions concerning the Constant Dynamic Consistent property, see Siniscalchi (2001, 2011), Ghirardato, Maccheroni and Marinacci (2008), and Pires (2002).

[25] Any other work discussed by GMMS that takes as primitive a pair of preferences relations is atemporal (See Nehring (2001, 2009), Rubinstein (1998), Mandler (2005), and Danan (2008), Kopylov (2009)). More recently, Giarlotta and Greco (2013), Karni and Vierø (2012), and Lehrer and Teper (2013) also assume as primitive a pair of binary relations, but without any investigation on the standard updating rules. Actually, Karni and Vierø (2012) and Lehrer and Teper (2013) study the case where the DM become aware of new information not considered before, and one of the primitive binary relations aims to capture the behavior after informational expansion.

are given by Bewley's preferences and that conditional subjective preferences are complete relations satisfying mild conditions, we obtain that Intertemporal Consistency and Intertemporal Default to Certainty constitute the axiomatic justification for the sequential consistency of maxmin preferences. Most notably, such conditions also implies that the full Bayesian updating rule is the only way to updating the *ex ante* set of priors objectively revealed which characterizes conditional subjective preferences. Our other results (Theorem 3 and Corollary 4) check the robustness of such conclusion. Those results provide different ways of obtaining full Bayesian updating as the role of revising beliefs in the model of objective and subjective rationality.

5 Appendix

Proof of Results in the Main Text

Proof of Theorem 1:

By simple computation we can show that if $\succsim_E^{\#}$ is a maxmin expected utility preference represented by $\left(u, C^E\right)$ then Intertemporal Consistency and Intertemporal Default to Certainty hold for the pairing $\left(\succsim^*, \succsim_E^{\#}\right)$.

For the converse, note that since the binary relation $\succsim^*$satisfies Basic Conditions, C-Completeness and Independence it is represented by a Bewley's unanimity rule, *i.e.,* there exist an affine utility index $u : X \to \mathbb{R}$ and a nonempty, convex, and closed set of priors $C \subset \Delta$ such that for all $f, g \in \mathcal{F}$

$$f \succsim^* g \Leftrightarrow \int_S u(f)\, dp \geq \int_S u(g)\, dp,\ \forall p \in C.$$

Given an arbitrary objectively relevant event $E \in \Sigma$, we note that $\succsim^*$ and $\succsim_E^{\#}$ are the same over constant acts. In fact, applying, respectively, the definition of an objective relevant event and Intertemporal Consistency we get that

$$x \succsim^* y \ \Rightarrow xEy \succsim^* y \Rightarrow \ x \succsim_E^{\#} y.$$

Now, again using the definition of an objective relevant event and then applying Intertemporal Default to Certainty we get that

$$x \succ^* y \ \Rightarrow x \succ^* yEx \Rightarrow \ x \succ_E^{\#} y.$$

Therefore, $\succsim^*$ and $\succsim_E^{\#}$ coincide on X , and the mapping $X \ni x \mapsto u(x)$ represents both preferences on X.

Note that Monotonicity of $\succsim^*$ and Intertemporal Consistency imply that $\succsim_E^{\#}$ satisfies Monotonicity on E: Let $f, g \in \mathcal{F}$ such that $f(s) \succsim_E^{\#} g(s)$ for any $s \in E$. We have that $h := fEg$ is such that $h(s) \succsim^* g(s)$ for all $s \in S$ and Monotonicity of $\succsim^*$ gives $fEg \succsim^* g$ and applying Intertemporal Consistency we get $f \succsim_E^{\#} g$.

Since $\succsim_E^{\#}$ satisfies Monotonicity and Mixture Continuity, for any act $f \in \mathcal{F}$ we can find $x_f \in X$ be the certainty equivalent of f w.r.t. $\succsim_E^{\#}$.

We note that if not $fEx_f \succsim^* x_f$ then by Intertemporal Default to Certainty we obtain $x_f \succ_E^{\#} f$. Hence, $fEx_f \succsim^* x_f$ and since it is represented by the unanimity rule w.r.t. (u, C), we obtain that for any $p \in C$

$$\begin{aligned} \int_S u\left(fEx_f\right) dp &\geq u\left(x_f\right) \Rightarrow \int_E u\left(f\right) dp + p\left(E^c\right) u\left(x_f\right) \geq u\left(x_f\right) \\ &\Rightarrow \frac{1}{p\left(E\right)} \int_E u\left(f\right) dp \geq u\left(x_f\right) \Rightarrow \int_E u\left(f\right) dp^E \geq u\left(x_f\right). \end{aligned}$$

That is, for all $q \in C^E$, $\int_E u\left(f\right) dq \geq u\left(x_f\right)$, which gives also that

$$u\left(x_f\right) \leq \min_{q \in C^E} \int_E u\left(f\right) dq.$$

If the strictly inequality holds then there exists $y \in X$ such that

$$u\left(x_f\right) < u\left(y\right) < \min_{q \in C^E} \int_E u\left(f\right) dq,$$

which implies that for all $p \in C$

$$u\left(x_f\right) < u\left(y\right) < \int_S u\left(fEy\right) dp,$$

that is

$$fEy \succsim^* y \text{ and } y \succ_E^{\#} x_f,$$

and by Intertemporal Consistency

$$f \succsim_E^{\#} y \text{ and } y \succ_E^{\#} x_f,$$

and since $\succsim_E^{\#}$ is a preorder we obtain $f \succ_E^{\#} x_f$, which is impossible. Hence, $\succsim_E^{\#}$ is a maxmin expected utility preference represented by $\left(u, C^E\right)$.

Proof of Proposition 2:

$\succsim_E^*$ *satisfies C-Completeness*: Given two consequences $x, y \in X$, we know that $\succsim^*$ satisfies C-Completeness, so w.l.o.g. assume that $x \succsim^* y$. By Monotonicity of $\succsim^*$ we get that $xEy \succsim^* y$ and Objective Dynamic Consistency gives that $x \succsim_E^* y$.

$\succsim_E^*$ *satisfies Monotonicity (on E):* Let $f, g \in \mathcal{F}$ such that $f\left(s\right) \succsim_E^* g\left(s\right)$ for all $s \in E$. Note that by Monotonicity of $\succsim^*$ we get that $h := fEg \succsim^* g$. Hence, Objective Dynamic Consistency gives that $f \succsim_E^* g$.

$\succsim_E^*$ *satisfies Independence:* For all acts $f, g, h \in \mathcal{F}$ and $\alpha \in (0, 1)$, by applying Objective Dynamic Consistency twice and also using the fact that $\succsim^*$satisfies Independence we get that

$$\begin{aligned} f &\succsim_E^* g \Leftrightarrow fEg \succsim^* g \Leftrightarrow \alpha fEg + (1-\alpha) h \succsim^* \alpha g + (1-\alpha) h \\ &\Leftrightarrow \left[\alpha f + (1-\alpha) h\right] E \left[\alpha g + (1-\alpha) h\right] \succsim^* \alpha g + (1-\alpha) h \\ &\Leftrightarrow \alpha f + (1-\alpha) h \succsim_E^* \alpha g + (1-\alpha) h. \end{aligned}$$

Next, we present a fundamental lemma that has some interesting consequences.

Lemma 5 *Let $\left\{\succsim^{\lambda}\right\}_{\lambda\in\Lambda}$ be a family of binary relations over $\mathcal{F}$ and $\left\{\succsim_E^{\lambda}\right\}_{\lambda\in\Lambda}$ a family of conditional preferences for a given $\succsim^{\lambda}$-relevant event E for all $\lambda \in \Lambda$. Assume that for all $\lambda \in \Lambda$ the pairing $\left(\succsim^{\lambda}, \succsim_E^{\lambda}\right)$ satisfies Ordinal Consistency and Dynamic Consistency, and also that $\succsim_E^{\lambda}$ satisfies Consequentialism. Define*

$$\succsim := \bigcap_{\lambda\in\Lambda} \succsim^{\lambda}$$

and consider a conditional preference $\succsim_E$ w.r.t. some $\succsim$-relevant event E. Under such conditions the following conditions are equivalent:

(i) $\succsim_E := \bigcap_{\lambda\in\Lambda} \succsim_E^{\lambda}$;

(ii) The pairing $(\succsim, \succsim_E)$ satisfies Ordinal Consistency and Dynamic Consistency, and $\succsim_E$ satisfies Consequentialism.

Proof of *(i) $\Rightarrow$ (ii): Since $f \sim_E fEg$ if, and only if, $f \sim_E^{\lambda} fEg$ for all $\lambda \in \Lambda$, Consequentialism for $\succsim_E$ is immediate from the fact that each $\succsim_E^{\lambda}$ satisfies Consequentialism. Ordinal Consistency ($\forall x, y \in X,\ x \succsim y \Leftrightarrow x \succsim_E y$) is also straightforward.*

Now, given two acts $f, g \in \mathcal{F}$ the definition of $\succsim$ and $\succsim_E$ and the fact that all pairs $\left(\succsim^{\lambda}, \succsim_E^{\lambda}\right)$ satisfy Dynamic Consistency give that

$$f \succsim_E g \Leftrightarrow f \succsim_E^{\lambda} g,\ \forall\lambda \in \Lambda \Leftrightarrow fEg \succsim^{\lambda} g,\ \forall\lambda \in \Lambda \Leftrightarrow fEg \succsim g.$$

Proof of *(ii) $\Rightarrow$ (i): Ordinal Consistency says that both relations coincides over constant acts. Now, by definition of $\succsim$ and Dynamic Consistency for the pairing $(\succsim, \succsim_E)$ and also for all pairing $\left(\succsim^{\lambda}, \succsim_E^{\lambda}\right)$, $\lambda \in \Lambda$, we get that for all $f, g \in \mathcal{F}$*

$$f \succsim_E g \Leftrightarrow fEg \succsim g \Leftrightarrow fEg \succsim^{\lambda} g\ \forall\lambda \in \Lambda \Leftrightarrow f \succsim_E^{\lambda} g\ \forall\lambda \in \Lambda.$$

Hence, $f \succsim_E g$ if, and only if, $f \succsim_E^{\lambda} g\ \forall\lambda \in \Lambda$.

Now, take an affine utility index u and a nonempty, closed and convex set of priors C and consider the family of subjective expected utility preferences $\{\succsim^q\}_{q\in C}$, where each $\succsim^q$ is the expected utility preference represented by (u, q). Note that the Bewley's preference, denoted by $\succsim^C$, represented by (u, C) satisfies

$$\succsim^C = \bigcap_{q\in C} \succsim^q .$$

Given a $\succsim^C$-relevant event E, denote by $\succsim_E^C$ the respective conditional preference as well as denote by $\{\succsim_E^q\}_{q\in C}$ the family of conditional subjective expected utility with $q \in C$. We note that each pairing $(\succsim^q, \succsim_E^q)$ satisfies Dynamic Consistency and $\succsim_E^q$ satisfies Consequentialism if, and only if, we have that $\succsim_E^q = \succsim^{q^E}$ (see, for instance, Proposition 6 in Hanany and Klibanoff (2007)). From our Lemma 5, we get the next result, which is essentially the content of our Theorem 3.

Corollary 6 *Given a Bewley preference $\succsim^C$ and another nonempty, closed, and convex set of priors $\widehat{C}$. The pairing $\left(\succsim^C, \succsim_E^{\widehat{C}}\right)$ satisfies Dynamic Consistency and $\succsim^{C^E}$ satisfies Consequentialism if, and only if, $\widehat{C} = C^E$.*

Proof: *Since $\succsim^C = \bigcap_{q \in C} \succsim^q$ and each pairing $(\succsim^q, \succsim_E^q)$ satisfies Dynamic Consistency and $\succsim_E^q$ satisfies Consequentialism, Lemma 5 gives us the desired result.*

Remark 7 *We note that the Lemma 5 can be applied to a broader class of dynamic stable preferences as characterized by Gummen and Savochkin (2012, 2013). They show that not only expected utility preferences are dynamic stable but also the particular case of variational preferences of Maccheroni, Marinacci and Rustichini (2006) given by the Hansen and Sargent (2001) preferences. The same holds for a special class of confidence preferences of Chateauneuf and Faro (2009) having a second-order expected utility representation as characterized by Grant, Polak, and Strzalecki (2009).*

Remark 8 *In order to obtain the useful property of countable additivity of the multiple priors set $C \subset \Delta^\sigma(q)$, $q \in \Delta^\sigma$, it is enough to impose over the* ex ante *objective preference $\succsim^*$ the Monotone Continuity axiom as proposed in Ghirardato, Maccheroni, and Marinacci (2004)*[26]*. By our results, given a $\succsim^*$-relevant event E, if $\succsim^*$ satisfies the Monotone Continuity axiom then the subjective preference $\succsim_E^\#$ has a monotone continuous multiple priors representation of Chateauneuf et al. (2005).*

Proof of Corollary 4 is a simple combination of our results and the GMMS result (Theorem 4, p. 762).

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