General Equilibrium Option Pricing under Counter-Cyclical Growth and Long-Run Risk

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Abstract

Put option prices are counter-cyclical. We build a general equilibrium model based on Duffie-Epstein preferences and Ak production function that delivers a model of put option prices that captures both time-series and cross-sectional properties of relative put option prices. When estimated with US aggregate consumption data and S&P 500 index options using Bayesian MCMC, we confirm our theory that agents have elasticity of intertemporal substitution greater than 1 which confirms the substitution effect, and put option prices reveal the underlying counter-cyclical economic state. The underlying economic dynamics, when combined with long-run risk nature of Duffie-Epstein preferences, can match the time-series and cross-section of US option prices with our theory.

1 Introduction

The time-series and cross-sectional properties of put option prices offer interesting insight for preferences of a rational agent in the underlying economy. The time-series evidence, presented in Figure 1, suggest counter-cyclical nature of put option prices, and suggestive of counter-cyclical risk-premia in the economy. The cross-sectional evidence of put option volatility smile resulting from higher prices of out-of-the-money (OTM) puts relative to Black-Scholes framework. We propose and estimate a general equilibrium model of put option prices where we rationalize the counter-cyclical nature of put option prices with the consumption decision of the agent - both endogenously determined. Furthermore, our theory can also explain the put option smile by generating a level of OTM put option prices that are observed in the data.

Figure 1 illustrates the counter-cyclical nature of put option prices. We plot the time-series of relative put option prices in the post 1987 crash period upto 2006. Relative put option prices are the prices of options (in the Figure at-the-money (ATM) and 5% OTM) divided through by the price of the underlying S&P 500 Index futures. We observe that in times of uncertainty, like September 11, 2001, the Russian Ruble crisis in 1998, Enron bankruptcy, etc. relative put option prices are very high. From a rational viewpoint, that is probably due to the fact that in times of bad economic news a rational agent is averse to a loss in his wealth in the future and thus is willing to pay a high put option price to protect his wealth. The phenomenon is most revealing in the biggest economic expansion cycle, classified by NBER, between March, 1991 to March, 2001. At the trough of that cycle, the economy was coming out of a recession and an agent expected the economy to expand. The agent was willing to invest and expected a wealth increase from the expanding ecnomy. This reduced the price of put options in the first stage of that business cycle when the economy was growing. The economy started to slow down in the second stage where it was also hit by bad economic news of the Russian Ruble crisis and the Microsoft anti-trust lawsuit. During these uncertain times, the agent expected a future wealth loss in the economy and his desire to protect his wealth raised prices of put options in uncertain times. Indeed, the business cycle peaked in March 2001, bringing in a contraction of the economy. Soon after, the September 11th crisis and the bursting of the dotcom bubble introduced new uncertainties in the economy where the agent worried about a wealth loss paid high put option prices to insure his wealth. As the economy started to recover and expand, the agent expected his wealth to increase and thus his desire to pay for high put option fell.

We abstract the above phenonmenon into a general equilibrium setting. In our model, a rational agent exhibits stochastic differential utility (SDU) of Duffie and Epstein (1992), which is a continuous-time extension of Kreps and Porteus (1978) utility functions. In this economy, there is one firm and the firm produces dividends via a linear production technology, also known as an Ak production function. In equilibrium, the agent consumes all of his dividends and invests his entire wealth in the firm. The Ak production function has two different shocks. One is a traditional transient shock that produces an iid shock to capital growth, and the other is a small persistant shock in the expected return of the production technology that is time-varying and follows a linear markov process. This persistant shock becomes the underlying time-varying productivity shock in the economy, and under SDU, creates long-run risk in the agent's consumption decision and generates counter-cyclical risk-premia.

The counter-cyclical time-series nature of option prices is explained through this long-run risk variable and SDU in a general equilibrium setting. In our setting, the optimal consumption to capital ratio of the agent is time-varying due to the stochastic long-run shock, and is extremely non-linear due to SDU. The consumption to capital ratio is decreasing in the long-run productivity shock for risk-averse agents whose elasticity of intertemporal substitution is greater than 1 giving rise to the substitution effect. This is the central result that drives all time-series phenomenon. When the growth rate is increasing, consumption (relative to capital) falls, and risk-premia from long-run risk decreases. The agent invests more and expects his wealth to increase in the future. Thus, he is not concerned about insuring himself from a wealth loss which decreases the price of put options. Precisely the opposite happens in bad economic states when the agent thinks that the return on the production technology should be low. As the long-run productivity shock decreases, the agent becomes more risk-averse for wealth loss. Due to the substitution effect, the agent consumes more (relative to capital) of his wealth in bad economic states. To induce the agent to invest in these bad time, the risk-premia resulting from long-run risk increases. The consumption motivation in bad states induces the agent to protect his wealth when he expects to suffer a wealth loss, thus increasing the price of put options to protect his wealth loss. Thus, we establish long-run risk from a persistant shock in the presence of recursive preferences to be the key determinants of the counter-cyclical properties of put option prices.

Next, we estimate our structural model with Bayesian MCMC. Our goal is to extract a time-series of the long-run risk variable to confirm the counter-cyclical nature and also to infer structural parameter estimates from the data of traded option prices and real consumption growth. The choice of Bayesian MCMC is due to its computational flexibility where there is high degree of non-linearity in the underlying model. We filter the growth rates by generalizing the principle of Bayesian Kalman Filters, and estimate the parameters through a Gibbs algorithm. The interesting outputs from option prices is the counter-cyclical nature of the underlying long-run state variable shown in Figure 6 and the parameter estimates of the structural model in Figures 2-5. We estimate the elasticity of intertemporal substitution from consumption growth to be between 1.4 and 1.6 thus confirming the substitution effect. We also estimate our risk aversion parameter to be between 6.11 and 8.97, which is a big improvement over CRRA preference which requires astronomical risk-aversion for matching assets with high risk-premia.

This paper adds to the growing body of literature on equilibrium option pricing by proposing a real business cycle setting with long-run risk and counter-cyclical premia. The paper closest to this is one by Benzoni, Collin-Dufresne and Goldstein (2005), who model a similar setting. However, there are several important differences. First, their setting is not general equilibrium because they do not endogenize the consumption decision of the agent. Second, they mention the need for counter-cyclical premia but their model doesn't generate it. The solution of their value function is exponentially affine in growth rates, which makes their long-run risk component in the pricing kernel a constant. Thus, while they are able to generate a steeper smile because of substantial long-run risk, they cannot generate business cycle effects because their market price of risk is not dependent on the growth rate.

Other equilibrium papers on option pricing such as Liu, Pan and Wang (2005) offers an explanation of OTM put options by considering model uncertainty a la Anderson, Hansen, and Sargent (2003). In their model, the agent exhibits uncertainty about the distribution of jumps in return. Since jumps are rare, the agent adds a rare-event premia to the jumps but not to the diffusion element which the agent has no uncertainty over because he observes it continuously. They can generate steeper volatility smirks if there is uncertainty surrounding the jump intensity. Brown and Jackwerth (2004) consider a setting where the pricing kernel of the agent has a stochastic component driven by a "momentum" state variable. Bates (2001) proposes a model where agents exhibit crash aversion to capture many observed facts from the options market. Constantinides, Jackwerth and Perrakis (2006) documents violation of stochastic dominance , i.e. non-existance of an agent whose pricing kernel is monotone decreasing in wealth, in out-of-the-money calls under incomplete market and generous transaction costs. The literature on real business cycle is too voluminous to be discussed in this setting. The closest paper is Ai(2007) which explores long-run risk in the presence of incomplete information and finds big welfare benefits to early resolution of uncertainty, i.e. when elasticity of intertemporal substitution greater than 1 for a risk-aversion agent.

Our paper also fits in to the body of literature that proposes alternative to the iid assumption of Black-Scholes by extending the underlying returns process to stochastic volatility (Heston (1993)) and stochastic volatility and jumps (Bates (1996)). Duffie, Pan and Singleton (2000) show how in affine setting a host of valuation problems can be analyzed through a class of transforms. In our case, aggregate wealth has a time-varying persistant growth rate that induces long-run risk. The time-varying growth rate follow a linear markov process making the underlying wealth and the premia associated with it within the affine class of models which can be solved using Fourier transforms.

The paper is organized as follows - section 2 discusses the GE setting and solves the social planner's problem, section 3 discusses asset pricing implications and derive option prices, section 4 discusses our MCMC algorithm and presents the empirical findings and section 5 concludes.

2 The Model

The representative agent is endowed with stochastic differential utility (SDU) of Duffie and Epstein (1992), which is the continuous time version of Epstein and Zin(1989). Utility is given by normalized aggregator $f(C, J)$ given by

$$f(C,J) = \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J\left[\left(\frac{C}{((1-\gamma)J)^{\frac{1}{1-\gamma}}}\right)^{1-\frac{1}{\psi}} - 1\right] \tag{1}$$

where C is the current consumption and J is the continuation utility given by the agent's value function. The parameters of this utility function are β - the discount rate for this infinitely lived agent, ψ - the elasticity of intertemporal substitution (EIS) and γ - risk-aversion. Unlike CRRA utility functions, SDU separates the close link between risk-aversion and intertemporal substitution. Under CRRA preferences, $\psi = \frac{1}{\gamma}$, whereas under SDU that tight relationship is broken. In fact, Schroder (1999) agents show early (late) resolution of uncertainty if $\gamma > 1/\psi$ ($\gamma < 1/\psi$).

2.1 Prices and Dividends

The asset market contains a default-free bond B_t which earns a risk-free rate r_t^f and a stock P_t which is the technology that converts capital into consumption good. Any other asset, like options, is available in zero-net supply.

Dividend flows/aggregate output are represented by y_t. In time period dt, firm pays dividends $y_t dt$. Returns on aggregate wealth - R_t, is characterized by the equilibrium return on stock plus the dividend-yield every period.

2.2 Output and Capital

We assume that output y_t can be produced by the capital stock K_t using an $A(\theta_t)k$ production function with stochastic production fluctuations given by

$$A(\theta_t) = \theta_t dt + \sigma_K dB_K \tag{2}$$

The expected return on the production technology - θ_t, itself is time-varying and follows a mean-reverting Ornstein-Uhlenbeck process of the form

$$d\theta_t = \delta(\bar{\theta} - \theta_t)dt + \sigma_\theta dB_\theta \tag{3}$$

In this economy, dB_K and dB_θ are correlated Brownian motion shocks with correlation ρ. σ_K and σ_θ represent a transient shock to production and returns to production respectively. Production shock consists of a transient shock σ_K due to the Brownian motion dB_K and a predictable shock θ_t, which in the presence of SDU creates long-run risk for the representative agent. The process of θ_t is known to the representative agent, unlike in Ai(2007), where the agent has uncertainty over this shock. δ and $\bar{\theta}$ represent the speed of mean reversion and the long-run average of this predictable shock.

Given the linear specification of the production function, capital formation in this economy is given by the usual law of motion

$$\begin{aligned} dK_t &= A(\theta_t)K_t - y_t dt \\ &= K_t[\theta_t dt + \sigma_K dB_K] - y_t dt \end{aligned}$$

Capital grows due to a production process that is linear in the level of the capital stock minus the output produced by the capital stock. Here, the stochastic production fluctuation $A(\theta)$ represents a production shock that is net of depreciation. Capital growth will be due to a transient shock σ_K in $A(\theta)$ and also due to a time-varying stochastic component θ_t which induces long-run risk.

In equilibrium, the agent consumes dividends $c_t^* = y_t^*$ and holds the stock. Since capital is the only factor of production, the return on the stock, which is the technology that converts capital to consumption good, is the same as the return on capital. Thus, $\frac{dP^*}{P^*} = \frac{dK^*}{K^*}$, and if $P_0 = K_0$ then $P_t^* = K_t^*$. Hence, return on aggregate wealth follows

$$\frac{dR^*}{R^*} = \frac{dP^*}{P^*} + \frac{y_t^*}{P^*}dt \tag{4}$$

$$= \frac{dK^*}{K^*} + \frac{y_t^*}{K^*} \tag{5}$$

2.3 Social planner's problem and Equilibrium Consumption

The First and Second Welfare Theorems apply directly, and thus competitive equilibrium allocations coincide with the Pareto allocations of the social planner. The social planner's problem is

$$\sup_{C_t} E_t \left[\int_t^{\infty} f(C_s, J_s) ds \right] \tag{6}$$

subject to

$$dK_t = K_t[\theta_t dt + \sigma_K dB_{Kt}] - C_t dt \tag{7}$$

where C_t is the aggregate consumption. The equilibrium condition - aggregate dividends equal aggregate consumption is already substituted in the constraint.

Propostion 1: The solution of the social planner's problem (6) is given by

$$J(\theta, K) = H(\theta)\frac{K^{1-\gamma}}{1-\gamma} \tag{8}$$

Optimal capital is given by

$$\frac{dK^*}{K^*} = \left[\theta_t - \beta^\psi H(\theta_t)^{\frac{1-\psi}{1-\gamma}}\right] dt + \sigma_K dB_{Kt} \tag{9}$$

and optimal consumption, $C^* = y^* = \beta^\psi H(\theta)^{\frac{1-\psi}{1-\gamma}} K$ where $H(\theta)$ is given by (27).

Proof: See Appendix A.

The ratio of optimal consumption to the capital stock is not constant, but proportional to a non-linear function of the expected growth rate of the economy. For $\psi = 1$, the consumption to capital ratio is constant. For $\psi \neq 1$, the nature of the function $H(\theta)$ is crucial. $H(\theta)$ is monotonically decreasing, which implies $H'(\theta)$ is negative. As such, for relatively risk-averse agents ($\gamma > 1$), the consumption-capital ratio is increasing in current state of productivity (θ_t) if the EIS, ψ, is less than 1, and decreasing for EIS greater than 1 ($\psi > 1$).

This also creates income or substitution effects across the business cycle. If the growth rate of productivity (θ_t) is increasing then the agent could either consume more (income effect) or substitute consumption for a later period by investing more (substitution effect). The course of action that the agent takes will be determined by the value of EIS. As discussed above, the agent increases his consumption (relative to capital) with higher growth rate if $\psi < 1$, thus showing income effect. If $\psi > 1$, the agent's consumption will fall (relative to capital) when he faces higher growth rate. The agent will substitute consumption for a later period and will invest more. Thus, if $\psi > 1$, the agent is willing to invest more under high growth rates and thus overall return premia is low. For decreasing growth rates and $\psi > 1$, we expect consumption to rise (relative to capital). Thus the return premia should be higher to give incentive to the agent to give up consumption and invest more. Consequently, the stochastic growth rates with $\psi \neq 1$ creates a rich business-cycle environment to study endogenously determined joint asset prices and consumption.

The above logic creates the motivation for what we observe in the put option market. We observe higher put option prices in bad economic times when the growth rates are decreasing. If the agent has EIS greater than 1, then in bad economic times (decreasing θ_t) the agent would like to invest less and consume more (relative to capital). Thus, the agent is very averse to a wealth loss (measured in consumption) and is willing to pay a higher price for a put option to protect his wealth. That is what we observe in Figure 1. Bad economic signals imply falling growth rates (θ_t) and we observe high relative put option prices in these periods. High put option prices are driven by the desire of the agent to protect his wealth in bad states of the world (decreasing θ_t) when he would like to consume more of his wealth (measured in consumption).

Applying Ito's Lemma to C^*, we get

$$\frac{dC^*}{C^*} = \mu_C dt + \sigma_K dB_K + \frac{1-\psi}{1-\gamma}\frac{H'(\theta)}{H(\theta)}\sigma_\theta dB_\theta \tag{10}$$

where

$$\mu_C = \theta - \beta^\psi H(\theta)^{\frac{1-\psi}{1-\gamma}} + \frac{1-\psi}{1-\gamma}\frac{H'(\theta)}{H(\theta)}\left(\delta(\bar{\theta}-\theta)+\sigma_K\sigma_\theta\rho\right) + \frac{\sigma_\theta^2}{2}\frac{1-\psi}{1-\gamma}\left(\frac{H''(\theta)}{H(\theta)} + \frac{\gamma-\psi}{1-\gamma}\frac{H'(\theta)^2}{H(\theta)^2}\right) \tag{11}$$

Notice, if $\psi = 1$, then consumption growth and capital growth would be the same, but that is not the case if $\psi \neq 1$. If $\psi \neq 1$, expected consumption growth (μ_C) is very different from expected capital growth in (9). Similarly, if $\psi = 1$, then the second component of consumption volatility also disappears. If $\psi \neq 1$, then the second volatility term contributes to long-run consumption risk. Interestingly, for $\psi > 1$ and $\gamma > 1$, the overall volatility of consumption growth could be *smaller* than volatility of capital growth even under the added long-run risk since $H'(\theta) < 0$. This result is potentially due to early resolution of uncertainty. We know that for Duffie-Epstein preferences, an agent will prefer early resolution of uncertainty if $\gamma\psi > 1$. If γ and ψ are both greater than 1, then that requirement is satisfied trivially resulting in lower consumption volatility than volatility of capital growth. The implication of the above is that long-run risk will add substantial risk-premia in order to match the time-series and cross-sectional nature of put option prices. It will produce counter-cyclical market prices of risk which is absolutely essential to match the counter-cyclical nature of put option prices in the time-series.

Aggregate stock price follows $\frac{dP_t^*}{P_t^*} = \frac{dK^*}{K^*}$ in (9) and thus the total returns to agregate wealth is

$$\frac{dR_t}{R_t} = \frac{dP_t^*}{P_t^*} + \frac{y_t^*}{P_t^*}dt = \theta_t dt + \sigma_K dB_K$$

Thus, equilbrium returns on aggregate wealth follow the same process as the production technology, $A(\theta_t)$. The predictable time-series properties of aggregate wealth is driven through by the predictable component of the technology shock, θ. Under a bad shock (decreasing θ) the agent is expected to undergo a wealth loss (measured in consumption). This is also the time where he would like to increase his consumption. Thus, put options written on aggregate wealth, R_t, mitigate the problem for the agent by guaranteeing a certain level of wealth. He is willing to pay a higher amount to protect his consumption in bad times, which is essentially the time-series result we are seeking. Now we explore the effect of this long-run shock on marginal utilities and asset prices.

3 Asset Pricing

In order to price assets in this setting, we need to compute the marginal utility of an agent with preferences and consumption choices given above. The marginal utility of an investor with SDU is provided in Duffie and Epstein (1992).

Proposition 2: Given equilbirum consumption in (10) from above and the utility function (1), the dynamics of the marginal utility process, Λ, of the investor is given by

$$\frac{d\Lambda}{\Lambda} = \frac{df_C}{f_C} + f_J dt \tag{12}$$

$$= r_t^f - \gamma\sigma_K dB_{Kt} + \frac{H'(\theta)}{H(\theta)}\sigma_\theta dB_{\theta t} \tag{13}$$

where r_t^f is the risk-free rate and is equal to

$$r_t^f = -\frac{\gamma(\gamma+1)}{2}\sigma_K^2 - \frac{\psi}{\psi-1}\left[\beta^\psi H(\theta)^{\frac{1-\psi}{1-\gamma}}\left(\frac{1}{\psi}-\gamma\right) - \beta(1-\gamma)\right] + \gamma\left[\theta - \beta^\psi H(\theta)^{\frac{1-\psi}{1-\gamma}}\right] - \frac{H'(\theta)}{H(\theta)}\left(\delta(\bar{\theta}-\theta) - \gamma\sigma_K\sigma_\theta\rho\right) - \frac{\sigma_\theta^2}{2}\frac{H''(\theta)}{H(\theta)} \tag{14}$$

and the equilibrium risk premia is given by

$$-Cov\left(\frac{d\Lambda}{\Lambda},\frac{dP}{P}\right) = -Cov\left(\frac{d\Lambda}{\Lambda},\frac{dK^*}{K^*}\right) \tag{15}$$

$$= \gamma\sigma_K^2 - \frac{H'(\theta)}{H(\theta)}\sigma_\theta\sigma_K\rho \tag{16}$$

Proof: See Appendix B.

There are two market prices of risk in the pricing kernel(13). The first is the traditional risk-premia term $\gamma\sigma_K$ which accounts for the transient shock to capital growth. The next risk-premia term is due to SDU and the stochastic productivity shock, θ_t, and accounts for long-run risk. The origin of this term is due to the risk posed by the persistant growth rate to shocks in consumption. Under SDU if the agent prefers early resolution of uncertainty, then a persistant growth path or big perturbations on the growth path (high σ_θ) will pose higher risks to the agent's utility function because it makes future growth rates reveal slowly and with more uncertainty. Thus for agents seeking early resolution of uncertainty, this persistant growth with transient shocks can induce substantial risk-premia as seen in Bansal and Yaron (2004), Ai (2007), etc. The SDU specification generates a very high market price of risk due to long-run risk, which exceeds the traditional transient shock term by an order of magnitude as will be shown below.

The solution presented here has a stochastic component to long-run risk which is substantial and will be the key ingredient in producing the high premia in bad times when the productivity state variable is low or negative (decreasing θ_t) which is needed in order to explain the time-series phenomenon of put option prices. A specific solution for $H(\theta)$ is linearizing the system around the long-run mean of the consumption-capital ratio proposed by Campbell, Chacko, Rodriguez and Viciera (2004) by first using a transform $H(\theta)^{\frac{1-\psi}{1-\gamma}} = I(\theta)^{-1}$ to re-formulate ODE (27). It provides a solution of the form

$$I(\theta) = \left(a + b\theta + \frac{c}{2}\theta^2\right) \tag{17}$$

where $\{a, b, c\}$ are functions of the underlying parameters of the state-space and preferences provided in the appendix. Therefore, the second market price of risk has a closed-form solution

$$\frac{H'(\theta)}{H(\theta)}\sigma_\theta = \frac{1-\gamma}{\psi-1}(b + c\theta)\sigma_\theta = (\tilde{b} + \tilde{c}\theta)\sigma_\theta$$

that is independent of ψ. Details are in the appendix. This is an important point of departure from Benzoni, Collin-Dufresne and Goldstein (2005). The long-run component of their solution is exponentially affine which produces a constant market price of risk and thus they have no time-series variation in their model.

Due to the quadratic solution of H here, the market price of risk is time-varying. The long-run risk premia has two components - a constant term and a stochastic term. Thus, there is a time-varying long-run premia across the business-cycle. As shown in Table 2, the long-run risk premium is substantial in size even at a low level of risk-aversion as long as growth rate is persistant with small transient shock σ_θ. The coefficient on θ, c, is strictly positive, so the persistant productivity shock offers counter-cyclical long-run risk premium which is necessary to explain the counter-cyclical pattern of put option prices.

3.1 Options

Other assets in the economy, as long as they are in zero net supply, will also be priced in this equilibrium. For example, let's consider an European put option written on the post-dividend aggregate wealth, R_t. The motivation for studying put options in this setting is the substitution effect that increases consumption (with respect to capital) in bad times. In bad economic states, (decreasing productivity θ_t) the agent prefers to consume his wealth if risk-aversion and EIS are both greater than 1. He is deterred away from investing and requires very high premia to give up consumption in bad states of the world. Since he would rather consume his wealth, he is willing to pay a higher price for insuring his wealth in bad times. Thus, the counter-cyclical nature of put option prices is likely due to counter-cyclical productivity shock which enters the agent's utility function as long-run risk. The price of the put option will be determined in equilibrium by the pricing kernel determined by (13) with the quadratic $H(\theta_t)$ function determined in the Appendix.

Consider a put option that has a payoff of $\max[X - R_T, 0]$ at time T which the agent is trying to price at time $t < T$. In this case, the put option insures the agent at time t from a loss in the wealth level below X. This put option offers *early resolution of uncertainty* regarding the agent's wealth at time T. As such, the specifications in his utility function ($\psi\gamma > 1$) that ensures substitution effect is also consistent with demand of assets like put options - all within a consistent general equilibrium framework.

First, the corresponding call option will be priced and then put-call parity will be used to back out the put option price. If the call option has a strike price of X, then it pays $\max[R_T - X, 0]$ unit of the consumption good at maturity T. Its price at time t, $G(K, \theta, \tau)$ satisfies

$$E_t[dG] = r_t^f G - GE_t\left[\frac{d\Lambda}{\Lambda}\frac{dG}{G}\right] \tag{18}$$

where $\tau = T - t$. The first term is the familiar risk-neutrality term and the second term is the adjustment for the risk. The adjustment for risk comes from the endogenous pricing kernel which reflects long-run risk and counter-cyclical risk-premia. Since the risk-premia is affine the technique of Bates(1996), Heston(1993) and Duffie, Pan and Singleton (2000) will directly apply and the semi-closed solution for option prices is given by

Proposition 3: The price of a call option that pays $\max[R_T - X, 0]$ at maturity is given by

$$G(R_t, \theta, t) = R_t P1(g_t, \theta_t, \tau) - XP2(g_t, \theta_t, \tau) \tag{19}$$

where $g_t = \ln(R_t)$ and $P1$ satisfies restriction (44). $P2$ satisfies an exactly analogous restriction also with a set of ODE's given by (45). Put option prices can be obtained through put-call parity.

Proof: See Appendix C.

Relative option prices can be obtained by dividing through the option price formula by X. Notice that pure levels of R_t or X do not enter into the above relationship except through the ratio X/R_t which has an interpretation of its own called moneyness. If $X/R_t = 1$, then the option is at-the-money and the cost of the put option is the insurance that the agent is willing to pay such that his wealth level at some point in the future T doesn't fall below the current wealth level at time t. If $X/R_t = 0.9$, then the put option protects the agent from more than a 10% drop in wealth level, i.e. it is 10% out-of-the-money (OTM). Using put-call parity, the corresponding relative put option price is

$$\frac{Put}{R_t} = \frac{X}{R_t}(Z(\theta_t, \tau) - P2(g_t, \theta_t, \tau)) - (1 - P1(g_t, \theta_t, \tau)) \tag{20}$$

where $Z(\theta_t, \tau)$ is the price of a zero-coupon bond that matures in τ-period and is endogenously determined given the risk-free rate r_t^f. Notice, that since we are only going to look at very short-maturity options Z should be very close to 1.

For call options, $P1$ has to satisfy the boundary condition that $P1(g_T, \theta_T, 0) = 1_{\{g_T \geq \ln(X_T)\}}$. Same holds true for $P2$. As such, $P1$ or $P2$ is the probability that the call option finishes in the money. As such, $1 - P1$ $(1 - P2)$ is the probability that the put option finishes in the money. In this long-run risk setting, the probability that the option finishes in the money is time-varying. It is determined by the stochastic long run risk term θ_t. Thus option prices reflect business cycle effects. Given a level of moneyness, if θ_t is high (expansions) then the expected wealth level is higher in the future making OTM put options worthless. The reverse is true if θ_t is low or negative (recessions). As the growth rate of productivity (θ_t) increases, under substitution effect the agent is willing to invest more and the corresponding long-run risk premia is low due to its counter-cyclical nature. Subsequently, the expected wealth level in the future is higher, which drives down OTM put option prices. Similarly, as the growth rate is decreasing, under the substitution effect consumption increases, the agent invests less as the corresponding long-run risk premia increases. The expected wealth level is lower which increases OTM put option prices.

Thus, the setting here contributes every ingredient necessary for studying a dynamic general equilibrium model of put option prices that is consistent with underlying consumption decision of the agent. The key endogenous results - consumption dynamics in (10) and put option prices in (20), when matched against the data should satisfy our hypotheses if we find that the consumption decision of the agent shows EIS greater than 1 which would confirm substitution effect, and the long-run productivity shocks (θ_t) inferred from put option prices are counter-cyclical. Notice that the EIS parameter ψ only shows up in consumption growth and due to the linearization scheme that produced the closed-form solution of $H(\theta)$ it does not appear in the market price of risk. Thus, we cannot infer EIS from option prices, but consumption only.

4 Empirical Methodology

The general equilibrium setting described above creates an endogenous state-space. Equilibrium consumption and relative put option prices can be written jointly as

$$\frac{dC^*}{C^*} = \mu_C(\theta_t, \lambda)dt + \sigma_K dB_K + \frac{1-\psi}{1-\gamma}(b + c\theta_t)\sigma_\theta dB_\theta \tag{21}$$

$$\frac{Put}{R_t} = \frac{X}{R}(Z(\theta_t, \tau) - P2) - (1 - P1) \tag{22}$$

$$d\theta_t = \delta(\bar{\theta} - \theta_t)dt + \sigma_\theta dB_\theta \tag{23}$$

where $\lambda = \{\delta, \bar{\theta}, \sigma_K, \sigma_\theta, \rho, \gamma, \psi, \beta, \mu\}$ is the parameter space of the underlying economy.

The first two equations are derived endogenously in this economy and they are functions of the underlying dynamics and preferences. The parameters that govern the system, λ, are embedded in every term on the right hand side of the first two equations along with the expected growth rate θ_t. Jointly, they define an endogenous state-space where the hidden state, θ_t, shows up non-linearly in the equilibrium quantities. Our goal in the empirical exercise is to obtain the parameter estimates of λ along with time-series estimates of θ.

4.1 Data

We use consumption data of Real Non-durables and Services obtained from the Bureau of Labor Statistics in annual frequency from 1929-2006. We divide the sum of Non-durables and Services by a population series to make it Per Capita Real Consumption of Non-durables and Services. This is the longest time-series of Real Consumption growth that is available in the US time-series. On the other hand, traded options data on the S&P 500 Futures is available daily from 1988-2006 and are interpolated at regular intervals of strike prices. A full panel of option prices (both call and puts) across strike prices and maturity are available daily. On the third Friday of every month, we take the put options with maturity of 28-35 days which is representative of the most liquid option for that month. Once we fix the maturity, we pick strike prices at an interval of 1, 0.98, 0.97, and 0.95, i.e. put options with $\frac{X}{R} = 1, 0.98, 0.97, 0.95$, which comprise our cross-section. Due to computational convenience we further re-sample and use only quarterly option prices. A time-series of the cross-section is presented in the top panel of Figure 4.

4.2 MCMC Methodology

Based on our general model, we run two different MCMC chains. In the first, our observations are option prices, taken quarterly. In the second, our observations are annual consumption growth rates. We do this because we have a very rich panel data on put options and we would like to see what option prices reveal about the underlying structural model. The consumption data (from unreported tests) is wholly uninformative for many of the parameters in λ. Thus, in order to get a good posterior estimate of the parameters, we first estimate them from options data. Then, we impose a strong prior restriction on the distribution of these parameters when we analyze the consumption data. Two parameters - ψ and β can only be inferred from consumption growth, so we put a heavy restriction on the other parameters obtained from option prices and estimate ψ and β from consumption growth.

In both cases, our MCMC takes the form:

$$\lambda \mid \theta \qquad \theta \mid \lambda$$

where λ denotes the set of underlying parameters and θ denotes the time-varying underlying state. For example, with the options data, $\lambda_o = (\sigma, \sigma_k, \sigma_\theta, \rho, \delta, \bar{\theta}, \gamma, \mu)$, and θ is the latent growth rates of each of the 76 quarters in our data. For consumption, the parameter space $\lambda_c = (\lambda_0, \psi, \beta)$ because ψ and β do not show up in option prices according to our theory.

To draw the states we use FFBS (forward-filtering, backward-sampling) (Fruhwirth-Schnatter (1994), Carter and Kohn (1996)). This is accomplished by simply discretizing the univariate state space. We believe this simple strategy is underutilized. For the consumption data the basic FFBS strategy is extended to allow for the dependence captured by ρ by letting the distribution of the observed consumption growth at time t depend on both the current state and one lag. For the put option prices, we augment the observation equation of relative put option prices with a pricing error term on the right to break the stochastic singularity resulting from 4 option prices and 1 state.

For the consumption data our obsevation equation is $p(y_t \mid \theta_t, \theta_{t-1}, \lambda_c)$ where y_t is annual consumption growth. For the options data our obsevation equation is $p(y_t \mid \theta_t, \lambda_o)$ where y_t consists of the four relative option prices corresponding to the four relative strike prices 1, .98, .97, .95. Neither of the observation densities is in closed form. They are computed using complex computational strategies and certainly involve difficult nonlinearities in λ. By discretizing the state we are able to simply draw the time series of latent states jointly with the actual concept behind FFBS described next.

Very briefly, the strategy is the following. If $\theta_t = \{x_0, \cdots, x_n\}$ is the support for θ_t, then the posterior distribution of θ at each point is

$$p(\theta_t = x_j | y_t, \lambda) = \sum_{\theta_{t-1}} p(\theta_t) p(\theta_t = x_j | \theta_{t-1}, \lambda) p(y_t | \theta_t = x_j, \lambda)$$

The first term is simply the prior at each point, the second term is the transition density obtained by (23), the third is the likelihood term of observing the data conditional on the state. Then simply integrate out θ_{t-1} to form the posterior distribution of $\theta_t | y_t$. The strategy is to go down the time-series of the data and generate those posterior distributions at each point. Go to the last data point and sample θ_T from its posterior, and simply come back to sample the states jointly

$$p(\theta_t = x_j | \theta_{t+1}, y_t) = \frac{p(\theta_{t+1} | \theta_t = x_j) p(\theta_t = x_j | y_t)}{\sum_{\theta_t} p(\theta_{t+1} | \theta_t) p(\theta_t | y_t)}$$

Notice, we didn't need the states to be linear. We could handle the non-linearities in the model very conveniently and draw the states jointly from the data by using the theory that created the endogenous state-space.

For the consumption data, this strategy works easily because computation of consumption growth and volatility from our theory is not computationally intensive. For the options data, the computation of the option prices is extremely time consuming. Nevertheless, we have found the strategy to work by carefully precomputing all required option prices at each possible pair of (θ_t, θ_{t-1}) given the grid and current λ.

In both cases, the draw of the state seems to work well and reasonably fast. To draw λ we use Metropolis-within-Gibbs. We again use a grid for each element of λ. Using a grid enables us to easily incoroporate support restrictions in our prior. For the consumption data we simply draw each component of λ_c, one at a time.

The draw of λ_o, the parameter vector for the options data is by far the most difficult draw. This is because computation of the options prices is expensive and the support of the posterior seems to have complex non-linear structure. Hence, our current strategy is to combine Metroplis draws of each parameter one at a time, with Metropolis draws of groups of parameters which we are highly dependent in the posterior.

We simulate our MCMC draw 30,000 times. At each iteration of the draw, i, we draw the $i-$th draw of the parameter space λ_o^i (λ_c^i) and the state θ_t^i. To make inference about our draws of parameters and state variable, we discard the first 20,000 for burn-in and use the rest 10,000 for our analysis described below.

4.3 Empirical Results

Our empirical goal is to show that the time-series of put option prices are counter-cyclical, i.e. in bad times (low θ) option prices are high and vice versa. Furthermore, we also need to show the substitution effect in the consumption choice of the agent, i.e. from the time-series of consumption growth infer whether EIS is greater than one. This joint hypothesis confirms our general equilibrium theory that the substitution effect drives put option prices

The fact that there is some underlying counter-cyclical phenomenon is present in put option prices is obvious from Figure 1. High prices are almost always associated with underlying uncertainty about the economy. The economic recovery that started with the NBER Trough on March 1991 led to low OTM put option prices. Prices continued to fall through the recovery eventually spiking again during the Russian Ruble crisis. Soon after, the economy entered the second phase of expansion eventually peaking in March 2001. The mild contraction in 2001 was accompanied by September 11, the Enron scandal and the bursting of the dotcom bubble in 2002. Throughout these economic uncertainties that signal poor ecnomic growth and negatively impacts the agent's wealth, put option prices were high. Following a series of Federal Reserve interest rate cuts from 2001 to 2003, the economy started to expand again resulting in high growth in the US economy. Since the highs of the early 2000s, the put option prices show a precipitous drop until 2006. The strength of our filtering mechanism is that we can take the panel of observed relative put option prices and invert the system to produce a full distribution of productivity growth rates at each point in the time-series. Thus, we can get a full posterior distribution of the state of productivity θ_t for each time t, and confirm our hypothesis regarding counter-cyclical premia.

First we analyze the options data and obtain posterior distribution of the parameters

λ_o and states θ_t. First, let us focus on the states. Figure 6 plots the time-series of the cross-section of relative put option prices on the top panel, and produces the implied time-series of the productivity growth rates θ_t in the bottom panel. The counter-cyclical pattern is obvious. In the same period (March, 1991 - March, 2001) of the longest NBER expansion, relative put option prices were low. The corresponding growth rates in the economy given in the bottom panel show strong economic growth rates in the economy. As the economy entered the contraction period following March 2001, and especially through the bad economic news of September 11 and the Enron accounting scandal, the agent's substitution effect (shown below) reveals his desire to protect his wealth that he will consume in bad states and hence the high prices for put options to protect his wealth loss. Thus, the agent's inference of the underlying economic state in these uncertain times falls from the highs of the 90s. Slowly, as the economy started to bounce back after repeated interest rate cuts from 2001-2003, the same substitution effect dictates that he would consume less of his wealth and invest more which would increase his future wealth. The agent infers that his future expected wealth is high and thus the price of put options to insure against a wealth drop falls again. Correspondingly, the underlying economic growth rates increase. The 2.5-97.5% quantile of the time-series of growth rates is shown in Figure 12. What is interesting is that across the business cycle, the full distribution of growth rates shift. From the highs in the mid-nineties to the lows in early 2000, the posterior distribution of the growth rates have moved significantly.

What should also be consistent from the counter-cyclical pattern of put prices above is that risk-premia should also be high in those periods when put option prices are high. The agent should consume more in those periods and invest less, thus the risk-premia should be high in bad growth rate states compared to good states precisely to give incentive to the agent to invest more of his wealth when he wants to consume. We take the posterior distribution of the parameter estimates λ_c that are summarized in Table 1, and compute the posterior distribution of risk-premia and the corresponding long-run risk-premia which accounts for the stochastic growth rate. Interestingly, Table 1 shows that the volatility of the transient shock σ_K is about 0.10 whereas the volatility of the long-run risk σ_θ is 0.01, such that transient shock is 10 times stronger than the corresponding shock in the long-run growth rate. However, long-run risk still dominates risk-premia as is shown in Table 2. It further shows the effect of long-run risk across the business cycle. Under a negative growth rate when the agent is expected to suffer a wealth loss, the risk-premia is considerable - 21% if the growth rate is -2% most of which is accounted for by the long-run risk 14%. Under high growth rates, the agent wants to invest more of his wealth thus the risk-premia is low. The agent no longer seeks premia for long-run risk and in fact the risk-premia changes sign. The agent expects his future wealth level is higher and is willing to take risks without compensation for risk. It is noteworthy to mention that the risk-aversion of the agent is relatively low in this setting. Table 1 shows that the median estimate of risk-aversion is 8.26 which is within the acceptable level, 10, that was recommended by Mehra and Prescott (1985).

The actual implied time-series of long-run risk is plotted in Figure 8. We take the posterior distribution of parameters and the state of productivity θ_t and produce the posterior distribution of long-run risk in the US economy as implied from the cross-section of option prices. Comparing Figures 8 and 6, it is obvious that states of high expected growth rates are associated with low long-run risk, whereas long-run risk is very high in times of high economic uncertainty. It is this stochastic nature of long-run risk that drives the time-series pattern of risk-premia. High risk-premia is required to induce the agent to give up consumption in bad times and invest his wealth. This high risk-premia is the compensation against wealth loss which is exactly what put options are supposed to insure against. As such, we establish stochastic long-run risk as the factor behind the counter-cyclical pattern of option prices.

Before we move on to option fits, we would like to point out the posterior estimate of $E\left[\ln \frac{C}{K}\right] = \mu$. The full posterior estimate of μ is shown in Figure 5. The posterior median from Table 1 is -3.35. As such, the long-run average consumption to capital ratio is roughly $e^{-3.35} \approx 3.5\%$. Hence, the panel of option prices suggest that in equilibrium real consumption should be about 3.5% (posterior distribution is between 1% to 6%) of total capital in the economy.

Option Fits: The time-series and cross-sectional fit of the panel of option prices is shown in Figures 9 and 11. We take our complete posterior distribution of parameters and states that we estimated from option prices and plug them back into the put option pricing formula (20). We plot our estimate of insample values of the time-series of the cross-section of option prices. Figure 9 shows the time-series fit of all 4 options that we consider across levels of moneyness $\frac{X}{R} = \{1, .98, .97, .95\}$. The posterior interval of option prices is marked by a line with the mean observed value at each point by a circle, whereas the true price is marked with an "x". We also produce a scatter plot of all the option prices with their observed counter-part in Figure 11. Both of these plots show that in-sample we are able to generate decent fit of the data with our theory.

We further produce a comparison of our dynamic theory with Black-Scholes in Figure 10. A noted problem with Black-Scholes option pricing theory is that Black-Scholes under-prices OTM put options. That problem manifests itself through the well-known put option smile, which is essentially a one-to-one mapping from higher prices to higher volatility of the underlying. Since Black-Scholes underprices OTM put options, the implied volatilities extracted from OTM put-options is higher than Black-Scholes, which is a constant and hence the same across the cross-section of put option prices. To compute Black-Scholes prices across the cross-section of moneyness, we pick interest rate of 3.4% (average annualized 1 month nominal interest rate from 1988-2006) and volatility of 15% (average annual S&P volatility from 1988-2006). In Figure 10 we plot the time-series average of observed option prices in black, and Black-Scholes values in green. As one can see, Black-Scholes underprices OTM put options considerably. At the same time, we plot our time-series average of the cross-section of fitted option prices which are plotted in Figures 9 and 11. The red line (our posterior average option prices) does a significantly better job than Black-Scholes does.

The basic ability of our model to explain OTM put option prices is shown in Table 3. The probability that a put option will finish in the money - (1-P2) is computed with the median of the posterior estimates of the parameters in Table 1 across different levels of moneyness and expected growth rates. An OTM put option has significantly higher probability to finish in the money under a bad expected growth rate state than a good state. For a 5% OTM option, the probability that a put option will finish in the money is 0.02 under expected growth rate $\theta_t = 0.05$ and 0.31 if the expected growth rate is $\theta_t = -0.05$. In fact, the probability increases monotonically from good expected growth rate state to bad ones across all level of moneyness.

Overall, what we have shown that if the agent exhibits substitution effect, then the cross-section and time-series of OTM put option prices are consistent with our theory of counter-cyclical premia delivered to us by DE preferences. Now, we look at consumption growth to confirm that agents exhibit substitution effect.

Consumption Growth: The time-series pattern of optimal consumption is given by (10). It is derived endogenously from the first order condition of the agent who exhibits DE preferences under an economy that has fluctuating economic growth. As mentioned above, the data of aggregate consumption simply isn't informative for our theory, so we use the rich option price data to infer the parameter estimates of our model. However, there are two parameters - EIS, ψ, and time-preference parameter, β, that cannot be inferred from option prices and we have to get their inference from aggregate consumption.

The way we proceed is that we impose a very tight prior restriction on the rest of the parameters that can be estimated from option prices. We impose heavy priors to restrict the parameters to have the same distribution as the posterior distribution from option prices. In other words, we take the posterior distribution from option prices and use them as strong priors for analyzing consumption growth.

Conditional on heavy restriction from the option parameters, λ_o, we estimate the posterior distribution of ψ and β and report them in Table 1. It shows that the posterior distribution of ψ has a median of 1.50 with a 95% posterior interval of $[1.40 - 1.58]$ and β has a median estimate of 0.01 with posterior interval between $[0.01 - 0.02]$. Our estimate of EIS is greater than one, which is what we need for substitution effect to prevail and the consumption data is very informative for that parameter as is evident from Figure 4.

So, how do we do in explaining the time-series of consumption growth? First of all, we have to re-estimate our growth rates θ_t for consumption growth because we have a different data frequency and time-period for real consumption growth. We take the posterior estimates of all parameters and the unobserved expected growth rates and produce the posterior distribution of μ_C - expected consumption growth. We plot consumption growth from the data and against expected consumption growth from our theory in Figure 7. The dotted lines reflect the inter-quartile range for μ_C at each time. Expected consumption growth is like a smoothed version of consumption growth and the posterior intervals cover the full range of the data.

This completes our proof. Our joint hypothesis that the counter-cyclical time-series pattern of option prices and consumption growth can be rationalized through substitution effect is hereby confirmed. A panel of option prices confirm the above. Then we take the posterior distribution of parameters that are consistent with observed option prices, turn them into prior restriction on consumption and produce posterior estimate of EIS to prove our claim.

5 Conclusion

A general equilibrium model of options prices is presented under Stochastic Differential Utility and long-run risk posed by a persistant expected growth rate shock. The model delivers risk-premia from long-run risk that is stochastic and counter-cyclical. The model is estimated using annual aggregate consumption data and put option prices from the S&P 500 Futures option prices.

Two features of the model are absolutely necessary in producing the desired effects. First, the substitution effect due to elasticity of intertemporal substitution greater than 1 that makes a risk-averse investor respond to falling growth rates by increasing consumption of wealth. The risk-averse agent who wants to consume his wealth would respond to falling growth rates by buying put options on aggregate wealth. Secondly, when the growth rate is falling the counter-cyclical nature of the premia kicks in. It generates high risk-premia which is consistent with high substitution effect, i.e. an agent who consumes his wealth in bad times is willing to pay a high price to insure it if the economy faces a bad shock. Our joint hypothesis - elasticity of intertemporal substitution greater than 1 and counter-cyclical premia, are supported empirically in the data.

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6 Appendix

6.1 A: Equilibrium

Solution of planner's problem

The HJB equation satisfying the social planner's problem is

$$0 = \max_{C} f(C, J) + J_\theta \delta(\bar{\theta} - \theta) + J_K(K\theta - C) + \frac{1}{2} J_{\theta\theta}\sigma_\theta^2 + \frac{1}{2} J_{KK} K^2 \sigma_K^2 + J_{K\theta} K \sigma_K \sigma_\theta \rho \quad (24)$$

The solution is

$$C^* = J_K^{-\psi} \beta^\psi \left[(1-\gamma)J\right]^{\frac{1-\psi\gamma}{1-\gamma}} \quad (25)$$

Plugging it back produces

$$\begin{aligned} \frac{(1-\gamma)J}{\psi - 1} \left[\frac{\beta^\psi}{J_K^{\psi-1}} [(1-\gamma)J]^{\frac{\gamma(1-\psi)}{1-\gamma}} - \psi\beta \right] + J_\theta \delta(\bar{\theta} - \theta) + \\ J_K K\theta + \frac{1}{2} J_{\theta\theta}\sigma_\theta^2 + \frac{1}{2} J_{KK} K^2 \sigma_K^2 + J_{K\theta} K \sigma_K \sigma_\theta \rho = 0 \end{aligned} \quad (26)$$

Notice that the value function J is homogeneous of degree $1-\gamma$ in K, and thus propose a solution $J = \frac{H(\theta)K^{1-\gamma}}{1-\gamma}$. That reduces the above PDE to an ODE

$$\begin{aligned} \frac{\beta}{\psi - 1} \left[\beta^{\psi-1} H(\theta)^{\frac{1-\psi}{1-\gamma}} - \psi \right] + \theta + \left[\frac{\delta(\bar{\theta} - \theta)}{1-\gamma} + \sigma_K \sigma_\theta \rho \right] \frac{H'(\theta)}{H(\theta)} + \\ \frac{1}{2} \frac{\sigma_\theta^2}{1-\gamma} \frac{H''(\theta)}{H(\theta)} - \frac{1}{2}\gamma\sigma_K^2 = 0 \end{aligned} \quad (27)$$

which means $C^* = y^* = \beta^\psi H(\theta)^{\frac{1-\psi}{1-\gamma}} K$.

6.2 B: Asset Pricing

The Pricing Kernel

Duffie-Epstein has shown that the state-price density for recursive preferences is given by

$$\Lambda_t = \exp\left[\int_o^t f_J(C_s, J_s)ds\right] f_c(C_t, J_t) \tag{28}$$

For the value function $J = \frac{H(\theta)K^{1-\gamma}}{1-\gamma}$,

$$f_C = K^{-\gamma}H(\theta) \tag{29}$$

$$f_J = \frac{\beta\psi}{\psi-1}\left[\beta^{\psi-1}H(\theta)^{\frac{1-\psi}{1-\gamma}}\left(\frac{1}{\psi}-\gamma\right)-(1-\gamma)\right] \tag{30}$$

So,

$$\frac{d\Lambda}{\Lambda} = \frac{df_C}{f_C} + f_J dt \tag{31}$$

$$= r_t^f dt - \gamma\sigma_K dB_{Kt} + \frac{H'(\theta)}{H(\theta)}\sigma_\theta dB_{\theta t} \tag{32}$$

where r_t^f is the risk-free rate and is equal to

$$r_t^f = -\frac{\gamma(\gamma+1)}{2}\sigma_K^2 - \frac{\psi}{\psi-1}\left[\beta^{\psi}H(\theta)^{\frac{1-\psi}{1-\gamma}}\left(\frac{1}{\psi}-\gamma\right)-\beta(1-\gamma)\right] + \gamma\left[\theta - \beta^{\psi}H(\theta)^{\frac{1-\psi}{1-\gamma}}\right] - \frac{H'(\theta)}{H(\theta)}\left(\delta(\bar{\theta}-\theta)-\gamma\sigma_K\sigma_\theta\rho\right) - \frac{\sigma_\theta^2}{2}\frac{H''(\theta)}{H(\theta)} \tag{33}$$

and the equilibrium risk premia is given by

$$
\begin{aligned}
-Cov\left(\frac{d\Lambda}{\Lambda},\frac{dP}{P}\right) &= -Cov\left(\frac{d\Lambda}{\Lambda},\frac{dK^*}{K^*}\right) \\
&= \gamma\sigma_K^2 - \frac{H'(\theta)}{H(\theta)}\sigma_\theta\sigma_K\rho
\end{aligned}
$$

An approximate solution

An approximate solution is available by linearizing the leading term of (27) around the long-run consumption-capital ratio. Let $H(\theta)^{\frac{1-\psi}{1-\gamma}} = I(\theta)^{-1}$. Notice that the optimal consumption condition implies

$$
\begin{aligned}
\frac{C^*}{K^*} &= e^{c-k} \\
&\approx e^{\mu}(1-\mu) + e^{\mu}(c-k) \\
\beta^{\psi} I(\theta)^{-1} &= e^{\mu}(1-\mu) + e^{\mu}\left(\psi\ln\beta - \ln I(\theta)\right)
\end{aligned}
$$

where $c = \ln C,\ k = \ln K,\ \mu = E(c-k)$. Substituting that in (27) simplifies the above expression to

$$
\begin{aligned}
\frac{1}{\psi-1}\left[h_0 + h_1(\psi\ln\beta - \ln I(\theta)) - \beta\psi\right] + \theta + \left[\frac{\delta(\bar{\theta}-\theta)}{1-\gamma} + \sigma_K\sigma_\theta\rho\right]\frac{1-\gamma}{\psi-1}\frac{I'(\theta)}{I(\theta)} + \\
\frac{\sigma_\theta^2}{2}\left[\frac{1}{\psi-1}\frac{I''(\theta)}{I(\theta)} - \frac{\gamma+\psi-2}{(\psi-1)^2}\frac{I'(\theta)^2}{I(\theta)^2}\right] - \frac{1}{2}\gamma\sigma_K^2 = 0
\end{aligned}
\tag{34}
$$

where $h_0 = h_1(1-\ln h_1)$ and $h_1 = \exp\mu$.The solution to the above is also $I(\theta) = \exp(a + b\theta + \frac{c}{2}\theta^2)$. Substituting that in the above and collecting terms, leaves three equations in three unknowns $\{a, b, c\}$ which can be solved sequentially.

The above system can be solved sequentially, first for c, then b and finally a. c has two solutions one of them is zero, but the other solution is utility maximizing and produces time-varying risk-premia. Taking that solution, $\{a, b, c\}$ solves

$$
\begin{aligned}
c &= \frac{\psi-1}{1-\gamma}\frac{2\delta+h_1}{\sigma_\theta^2} \\
b &= \frac{1-\psi}{1-\gamma}\frac{1}{\delta\sigma_\theta^2}\left((1-\gamma)\sigma_\theta^2+(h_1+2\delta)(\delta\bar{\theta}+(1-\gamma)\sigma_K\sigma_\theta\rho)\right) \\
a &= \frac{1}{h_1}\left[-\frac{(\psi-1)\gamma\sigma_K^2}{2}+\frac{1}{2}\sigma_\theta^2\left(C+\frac{1-\gamma}{\psi-1}B^2\right)+(\delta\bar{\theta}+(1-\gamma)\sigma_K\sigma_\theta\rho)B+h_1\psi\ln\beta+h_0-\beta\psi\right]
\end{aligned}
$$

The risk-premia is given by $\tilde{r}_t^P = \gamma\sigma_K^2 - \frac{1-\gamma}{\psi-1}\frac{I'(\theta)}{I(\theta)}$ which simplifies to

$$\tilde{r}_t^P = \gamma\sigma_K^2 - \tilde{b}\sigma_K\sigma_\theta\rho - \tilde{c}\sigma_K\sigma_\theta\rho\theta_t \tag{35}$$

where $\tilde{b} = \frac{1-\gamma}{\psi-1}b$ and $\tilde{c} = \frac{1-\gamma}{\psi-1}c$. The risk-premia is primarily determined by the risk-aversion, whereas the EIS impacts the risk-free rate. The risk-free rate is determined by

$$r_t^f = \left[\tilde{b}\sigma_K\sigma_\theta\rho - \gamma\sigma_K^2\right] + (1+\tilde{c}\sigma_K\sigma_\theta\rho)\,\theta_t \tag{36}$$

6.3 C: Equilibrium Options pricing

Let's start with the stochastic processes satisfying the optimal return(post-dividend) on the capital stock and expected return on the technology. (The *'s are dropped from the optimal quantities for readibility).

$$\begin{aligned}\frac{dR}{R} &= \theta_t dt + \sigma_K dB_K \\ d\theta_t &= \delta\left(\bar{\theta} - \theta_t\right) dt + \sigma_\theta dB_\theta\end{aligned}$$

Any derivative $G(R_t, \theta_t, \tau)$ that matures at $\tau = T - t$ satisfies the equilibrium condition

$$E_t[dG] = r_t^f G - G E_t\left[\frac{d\Lambda}{\Lambda}\frac{dG}{G}\right] \tag{37}$$

For an European style call option, the boundary condition must satisfy $G(R_T, \theta, \tau = 0) = \max[R_T - X, 0]$ if X is the strike price of the put option.

Define a variable $g_t = \ln(R_t)$. So, the process for g_t is

$$dg = \left[\theta_t - \frac{1}{2}\sigma_K^2\right] dt + \sigma_K dB_{Kt}$$

Applying Ito's Lemma to (37),

$$\begin{aligned}G_R R\theta + G_\theta \delta\left(\bar{\theta} - \theta\right) + \frac{1}{2}\left[G_{RR}R^2\sigma_K^2 + G_{\theta\theta}\sigma_\theta^2\right] + G_{R\theta}R\sigma_K\sigma_\theta\rho - G_\tau = \\ G r_t^f - G_R R\left((\tilde{b} + \tilde{c}\theta)\sigma_K\sigma_\theta\rho - \gamma\sigma_K^2\right) - G_\theta\left((\tilde{b} + \tilde{c}\theta)\sigma_\theta^2 - \gamma\sigma_K\sigma_\theta\rho\right)\end{aligned}$$

Notice that in the PDE for traditional option pricing, the expected return on stocks - time-varying or not - does not appear, because under the risk-neutral measure the stock simply grows at the risk-free rate. In this case, the expected return on the technology explicitly comes in through both the return dynamics and the marginal utility process. Collecting terms, one could write the above in the traditional risk-neutral way by accounting for the market price of risk.

$$G_R R\left(\theta - (\gamma\sigma_K^2 - (\tilde{b}+\tilde{c}\theta)\sigma_K\sigma_\theta\rho)\right) + G_\theta\left(\delta(\bar{\theta}-\theta) - \left(\gamma\sigma_K\sigma_\theta\rho - (\tilde{b}+\tilde{c}\theta)\sigma_\theta^2\right)\right) + \frac{1}{2}\left[G_{RR}R^2\sigma_K^2 + G_{\theta\theta}\sigma_\theta^2\right] + G_{R\theta}R\sigma_K\sigma_\theta\rho - G_\tau = Gr_t^f \tag{38}$$

Guess a solution similar to Black-Scholes' model of the form

$$G(R,\theta,t) = RP1(g,\theta,\tau) - XP2(g,\theta,\tau) \tag{39}$$

where $\tau = T - t$ is the time to maturity of the option. Substituting the solution (39) into the above PDE (38) reduces to,

$$RA_t \cdot P1(g,\theta,\tau) - XA_t \cdot P2(g,\theta,\tau) = 0 \tag{40}$$

where the operator A_t is a second order differential operator. This relationship will hold if the resulting diffferential equations are both equal to zero. The functional form of the PDE for $P1$ is

$$P1_g\left(r_t^f + \frac{\sigma_K^2}{2}\right) + P1_\theta\left(\delta(\bar{\theta}-\theta) + (1-\gamma)\sigma_K\sigma_\theta\rho + (\tilde{b}+\tilde{c}\theta)\sigma_\theta^2\right) + P1_{gg}\frac{\sigma_K^2}{2} + P1_{\theta\theta}\frac{\sigma_\theta^2}{2} + P1_{g\theta}\sigma_K\sigma_\theta\rho - P1_\tau = 0 \tag{41}$$

and for $P2$ is

$$P2_g\left(r_t^f - \frac{\sigma_K^2}{2}\right) + P2_\theta\left(\delta(\bar{\theta}-\theta) - \gamma\sigma_K\sigma_\theta\rho + (\tilde{b}+\tilde{c}\theta)\sigma_\theta^2\right) + P2_{gg}\frac{\sigma_K^2}{2} + P2_{\theta\theta}\frac{\sigma_\theta^2}{2} + P2_{g\theta}\sigma_K\sigma_\theta\rho - P2_\tau - r_t^f P2 = 0 \tag{42}$$

The existence of the above map depends on the continuity and boundedness of the function which will be assumed through the standard restrictions of the underlying markov process. The uniqueness of the solution depends on the convergence to the appropriate boundary condition which is now addressed.

At expiry, i.e. at $\tau = 0$, the boundary condition for $P1$ is $P1(g_T, \theta_T, 0) = 1_{\{g_T \geq \ln(X_T)\}}$. (The same logic applies for $P2$ and is omitted). It can be shown that $P1$ is the conditional probability that the option expires in the money, i.e.

$$P1(g, \theta, 0) = \Pr(g_T \geq \ln(X) | g_t, \theta_t)$$

This probability is not available in a tractable form due to the discontinuity of the boundary condition. However, the characteristic function of the probability distribution of g_T can be obtained in closed-form and that can be inverted to obtain the probability density of g_T. Assume that another function $f(g, \theta, \tau)$ satisfies (41) with the boundary condition $f(g_T, \theta_T, 0) = \exp(i\phi g_T)$, which is exactly the characteristic function I am seeking. The solution is readily available to be

$$f(g, \theta, \tau) = \exp\left(D(\tau) + E(\tau)\theta + i\phi g\right)$$

where D, E and F satisfy

$$\begin{aligned} D'(\tau) &= i\phi\left(\tilde{b}\sigma_K\sigma_\theta\rho - \gamma\sigma_K^2 + \frac{\sigma_K^2}{2} + E(\tau)\sigma_K\sigma_\theta\rho\right) - \frac{\phi^2\sigma_K^2}{2} + \\ &\quad E(\tau)(\delta\bar{\theta} + (1-\gamma)\sigma_K\sigma_\theta\rho + \tilde{b}\sigma_\theta^2) + E^2(\tau)\frac{\sigma_\theta^2}{2} \\ E'(\tau) &= i\phi\left(1 + \tilde{c}\sigma_K\sigma_\theta\rho\right) + E(\tau)(\tilde{c}\sigma_\theta^2 - \delta) \end{aligned}$$

with the initial condition that $D(0) = E(0) = 0$.

Now $P1$ can be easily obtained by fourier-inverting the characteristic function

$$P1(g,\theta,0) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{i\phi(g-\ln(X))}}{i\phi}\right]d\phi \tag{43}$$

This is the desired probability needed at the boundary when the option expires. $P1$ satisfies the above expression for all maturity τ, i.e.

$$P1(g,\theta,\tau) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{i\phi(g-\ln(X))+D(\tau)+E(\tau)\theta}}{i\phi}\right]d\phi \tag{44}$$

For $P2$ the characteristic function has the same form, except the ODE's solve

$$\begin{aligned}
D'(\tau) &= i\phi\left(\tilde{b}\sigma_K\sigma_\theta\rho - \gamma\sigma_K^2 - \frac{\sigma_K^2}{2} + E(\tau)\sigma_K\sigma_\theta\rho\right) + E^2(\tau)\frac{\sigma_\theta^2}{2} && (45)\\
&\quad + E(\tau)(\delta\bar{\theta} - \gamma\sigma_K\sigma_\theta\rho + \tilde{b}\sigma_\theta^2) - \left(\tilde{b}\sigma_K\sigma_\theta\rho - \gamma\sigma_K^2 + \frac{\phi^2\sigma_K^2}{2}\right) && (46)\\
E'(\tau) &= (i\phi - 1)(1 + \tilde{c}\sigma_K\sigma_\theta\rho) + E(\tau)(\tilde{c}\sigma_\theta^2 - \delta) && (47)
\end{aligned}$$

Table 1: This table reports the parameter estimates of the model by using Metropolis within Gibbs algorithm. We run the Gibbs sampler for 30,000 iterations and the first 20,000 are dropped for burn-in. The 10,000 remaining are used for all inference purposes. The data for relative put option prices are from put options on the S&P500 Future Index. We choose monthly option price (on the third Friday of every month) with 4 strike prices - ATM, 2% OTM, 3% OTM and 5% OTM with maturity 28-35 days. Data period for options is 1988-2006. The consumption data is annual and comes from BLS of non-durables and services and is deflated by the GDP deflator to convert into real consumption and further divided through by a population series to make it per capital real consumption. Data period is 1929-2006. All quantities rates are reported annually.

	Options		
	Posterior Median	2.5-97.5 quantile	
σ_K	0.10	0.09	0.11
σ_θ	0.01	0.01	0.02
ρ	0.92	0.83	0.95
δ	0.31	0.26	0.35
$\bar{\theta}$	0.04	0.03	0.05
μ	-3.35	-4.49	-3.01
γ	8.26	6.11	8.97
Pricing Error (σ)	0.02		
	Consumption		
ψ	1.50	1.40	1.58
β	0.01	0.01	0.02

Table 2: The following table reports the long-run risk and total premia in high or low states of θ_t. We take the full posterior estimates of parameters reported in Table 1 and evaluate the posterior estimate of risk-premia given in (16) along with the long-run risk component of the risk-premia given by (35) at different levels of θ_t. The median of the posterior estimate of risk-premia and long-run risk is reported here at different levels of θ_t.

θ	Long-Run Risk	Risk-Premia
-0.05	0.28	0.35
-0.04	0.21	0.28
-0.02	0.13	0.21
-0.01	0.06	0.14
0.01	0.00	0.07
0.02	-0.07	0.00
0.04	-0.15	-0.06
0.05	-0.22	-0.14

Table 3: This table shows the probability that a put option finishes in the money $(1 - P2)$ under high and low states of θ_t. We take the median posterior parameter estimates from the data and compute the posterior probability $(1 - P2)$ that a put option finishes in the money under different level of moneyness. The maturity has been fixed to 1 month for all options. The system of ODE's that $P2$ satisfies is given by (45).

$\frac{X}{R}$	θ							
	-0.05	-0.04	-0.02	-0.01	0.01	0.02	0.04	0.05
0.95	0.31	0.25	0.14	0.10	0.05	0.03	0.02	0.02
0.97	0.61	0.52	0.37	0.30	0.18	0.13	0.07	0.05
0.98	0.74	0.67	0.51	0.43	0.28	0.22	0.13	0.10
1.00	0.92	0.88	0.77	0.71	0.55	0.47	0.32	0.26

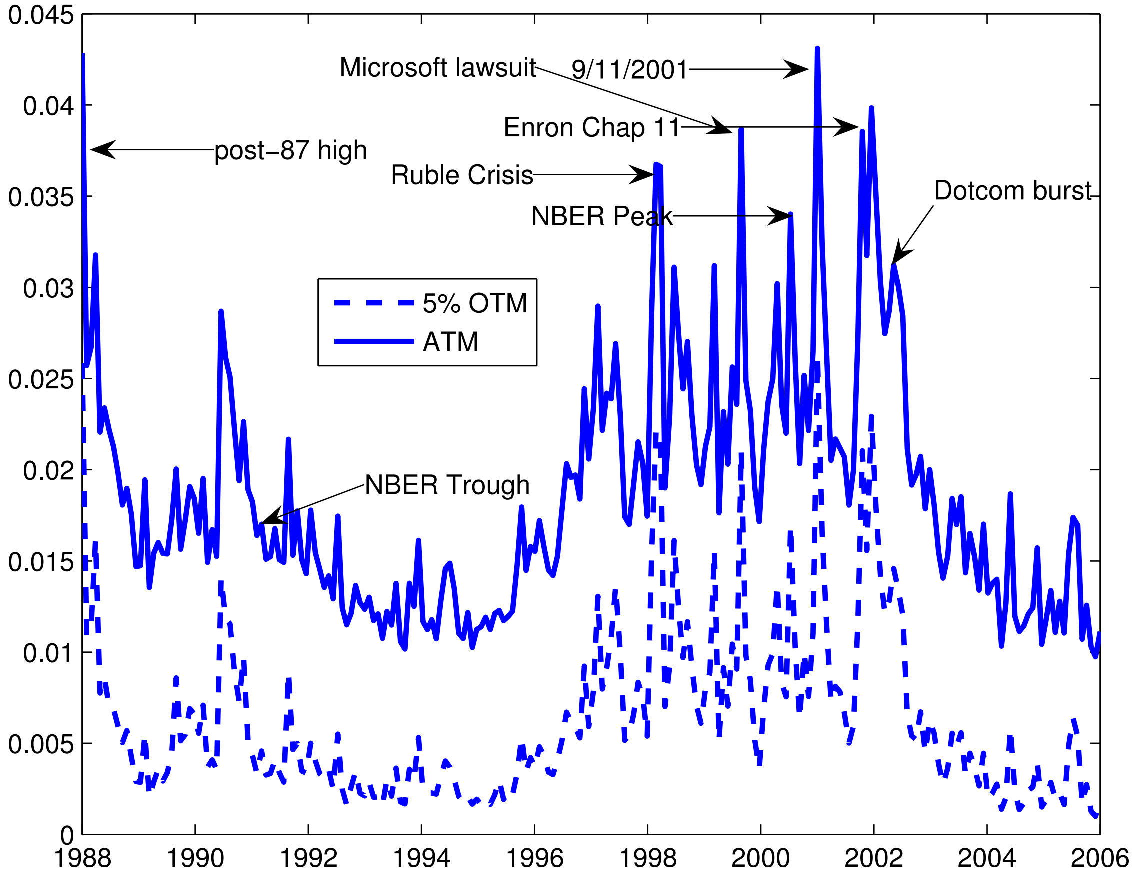

Figure 1: *The Time-Series of Relative Option Prices. This is a time-series plot of put-option prices divided by the price of the underlying S&P 500 Futures for options that are at-the-money and 5% out-of-the-money, with maturity of 28-35 days. The data above is of monthly frequency and the relative price on the third Friday of every month is chosen as the representative relative price for that month when the 28-35 day maturity options are most liquid.*

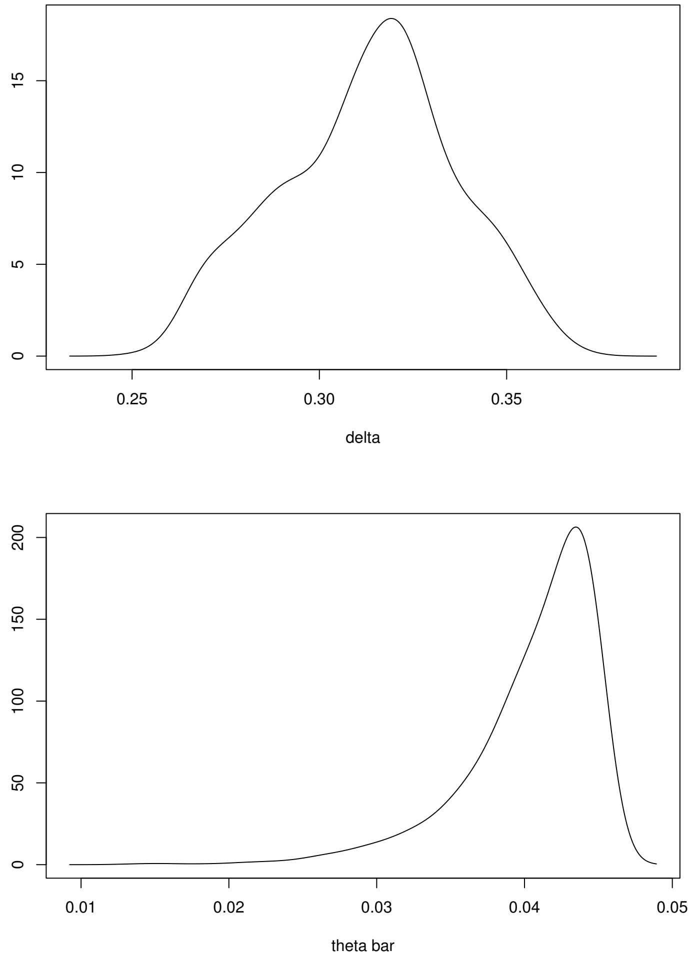

Figure 2: Posterior Distribution obtained of δ - the parameter of mean-reversion of θ, and $\bar{\theta}$ the long-run average of the growth rates θ. Both are obtained from option prices.

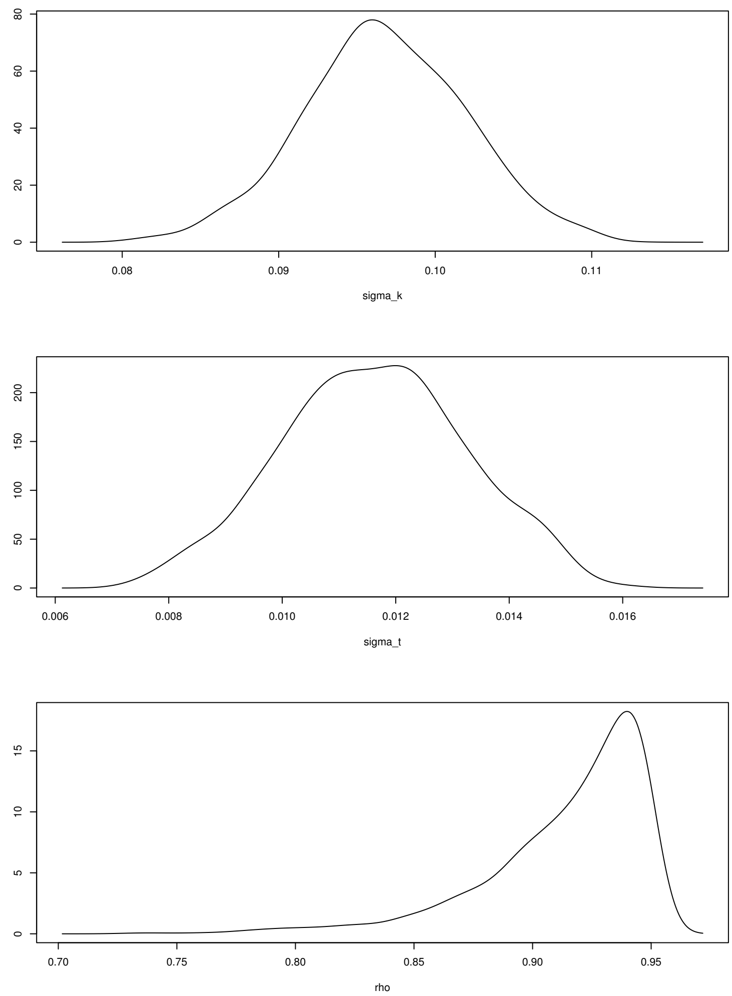

Figure 3: Posterior Distribution of the volatility parameters - σ_K, the volatility of capital growth, σ_θ, the volatility of the growth rates, and ρ the contemporaneous correlation between the shock in the shock in capital growth with expected growth rate. All three are obtained from option prices.

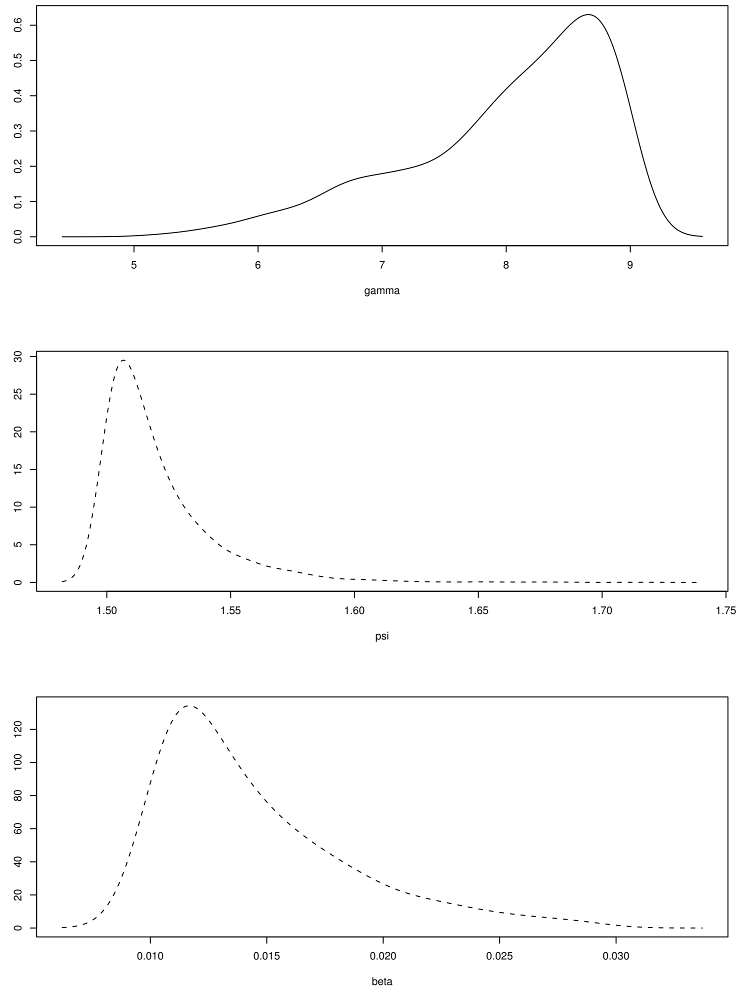

Figure 4: Posterior Distribution of Utility parameters - γ, risk aversion, ψ, elasticity of intertemporal substitution, and β, time-preference parameter. γ is obtained from option prices, whereas ψ and β are obtained from aggregate consumption.

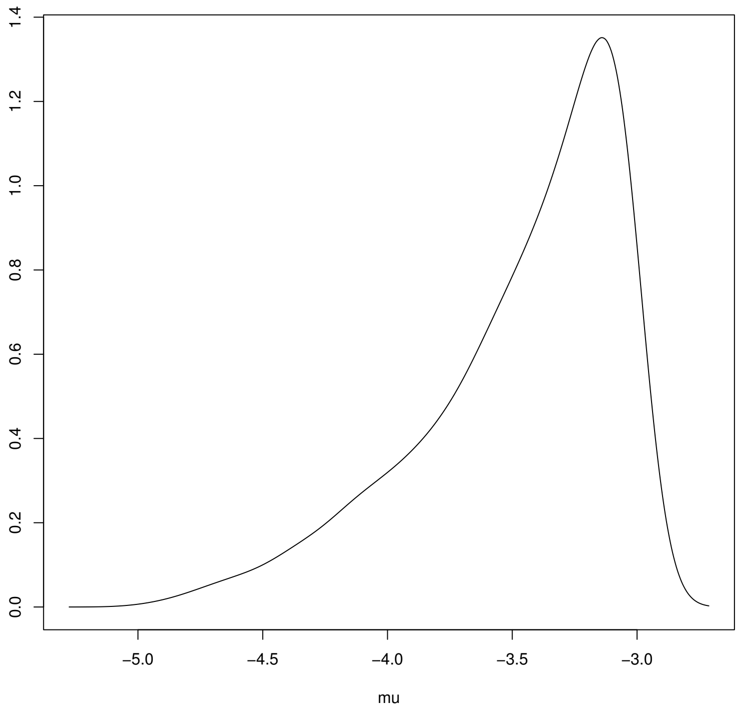

Figure 5: Posterior Distribution of μ - the long-run average of the log-consumption to capital ratio obtained from option prices.

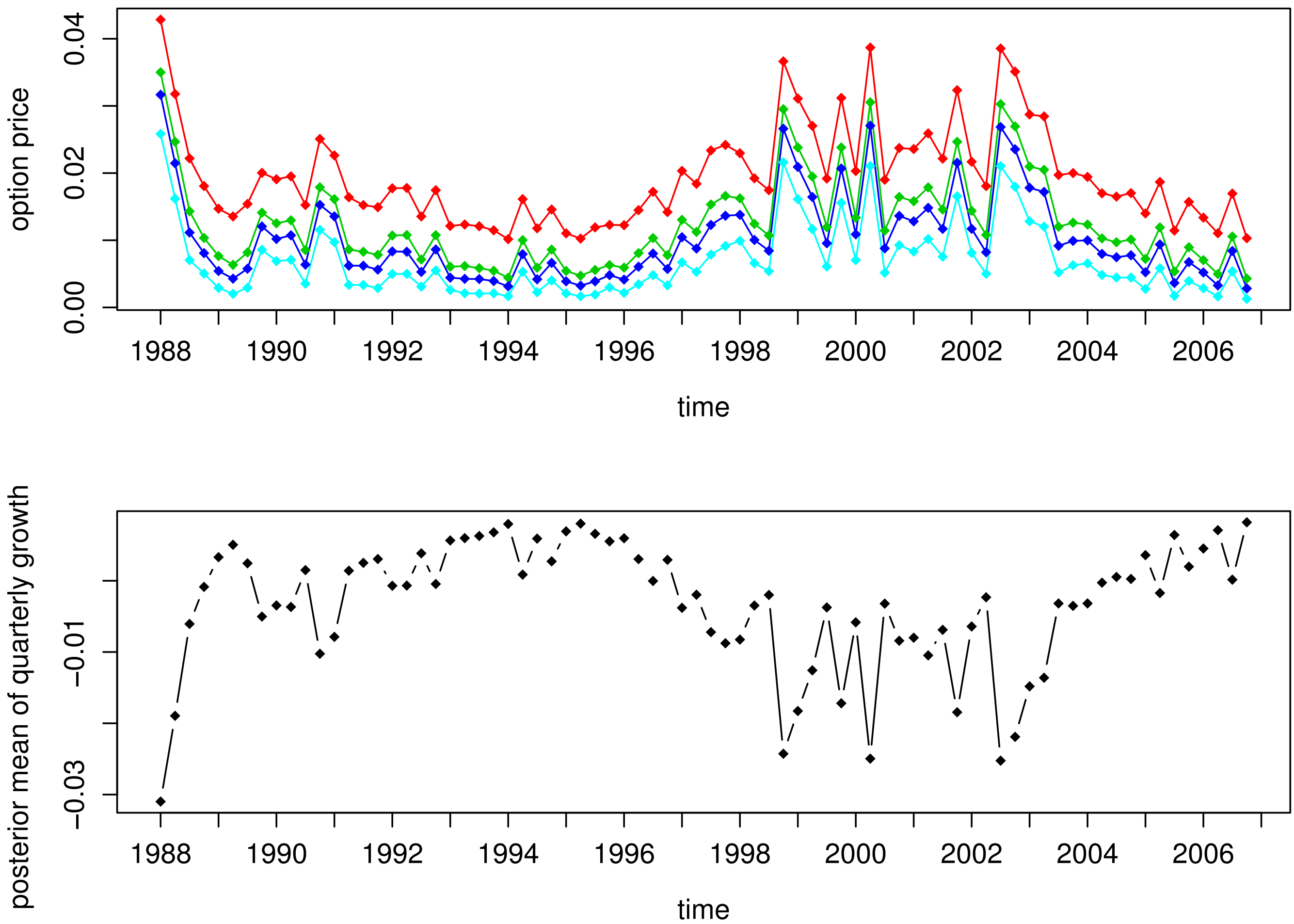

Figure 6: The top panel shows the time-series plot of the four option prices. The bottom panel shows the posterior mean of the quarterly growth rates that are filtered from the option prices.

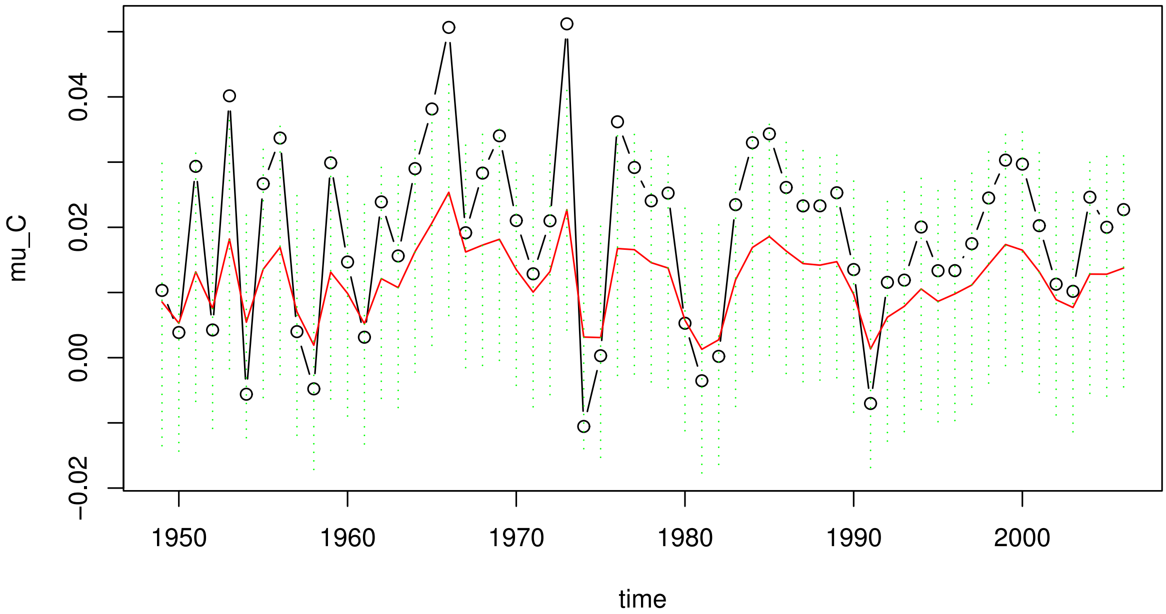

Figure 7: The black circles are the data points for aggregate consumption growth. The red line through the consumption data is the simulated posterior mean of consumption growth (μ_C) according to the model and the green dots are the inter-quartile range of the time-series of μ_C.

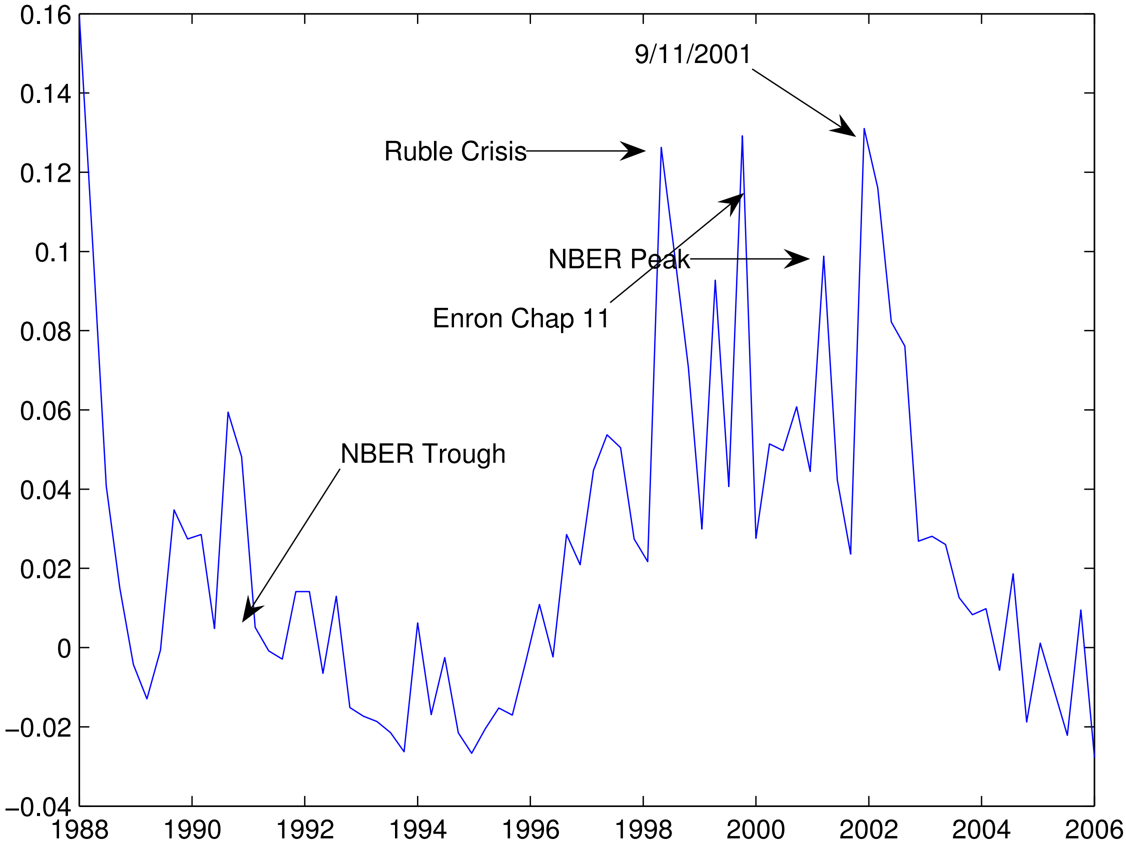

Figure 8: Time-Series of long-run risk estimated from option prices. Using the posterior distribution of parameters and the state, we compute the time-series of long-run risk given by $-\frac{H'(\theta)}{H(\theta)}\sigma_K\sigma_\theta\rho = -(\tilde{b}+\tilde{c}\theta)\sigma_K\sigma_\theta\rho$. The median time-series estimate is presented in the above plot.

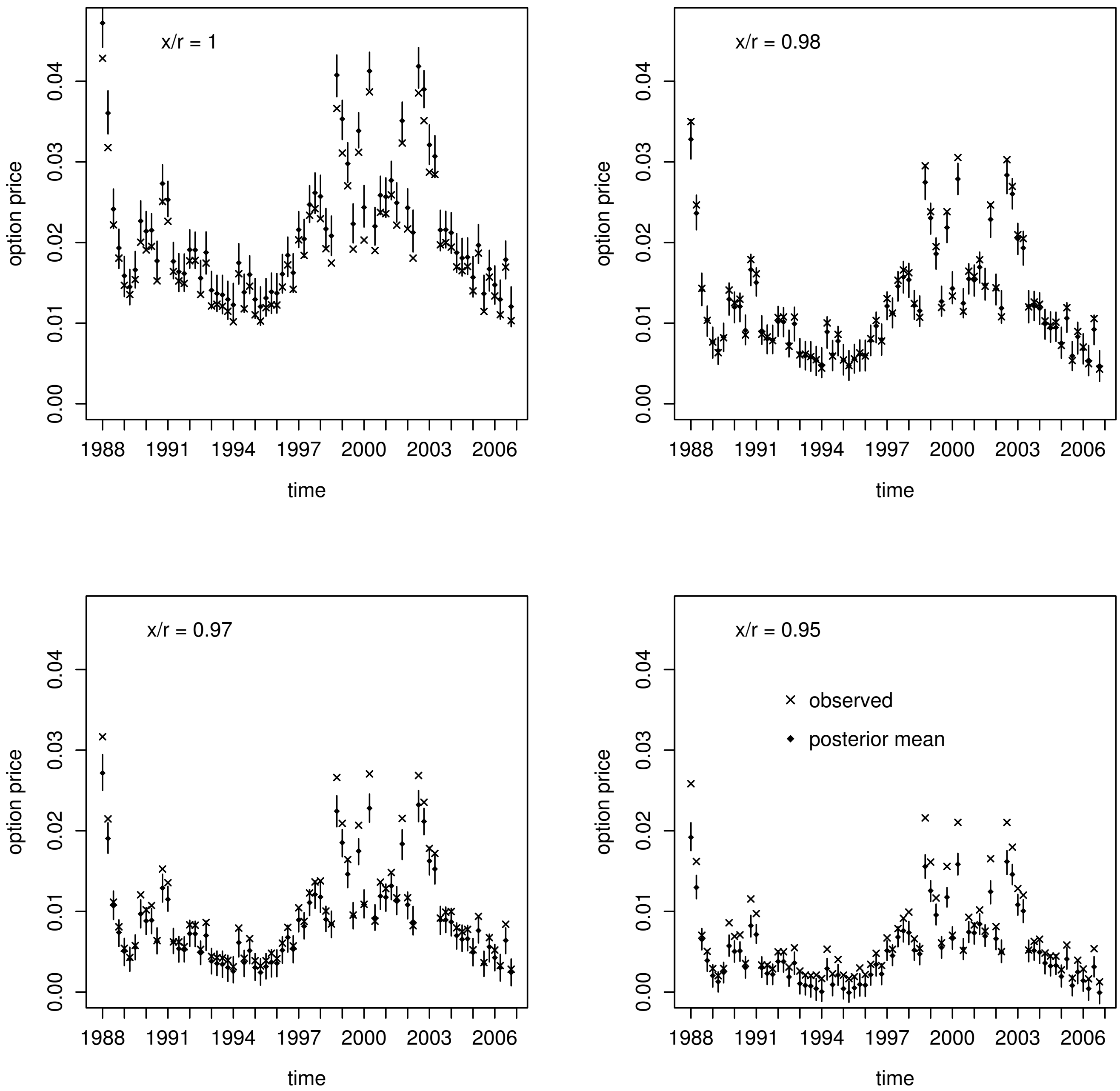

Figure 9: Time-series fit for Options. $\frac{X}{R} = 1$ implies ATM and $\frac{X}{R} = .95$ implies 5% OTM.The posterior parameter draws along with the posterior distribution of the states is used to compute the distribution of option prices at each point. The mean of the distribution at each point is marked with a circle, whereas the true value is marked by an x.

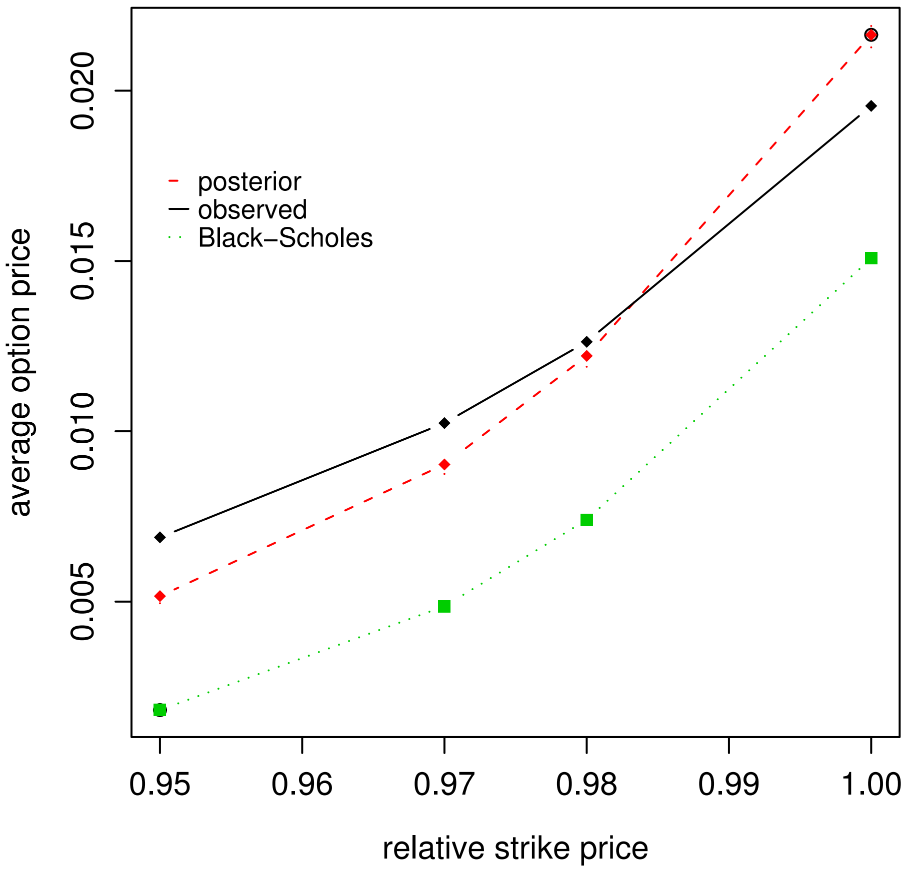

Figure 10: Cross-Sectional Fit for Put Options. The Black-Scholes line represents Black-Scholes prices with $r = .0396$(average nominal 1 month annualized interest rate from 1988-2006) and $\sigma = .15$(average annual volatility from 1988-2006)

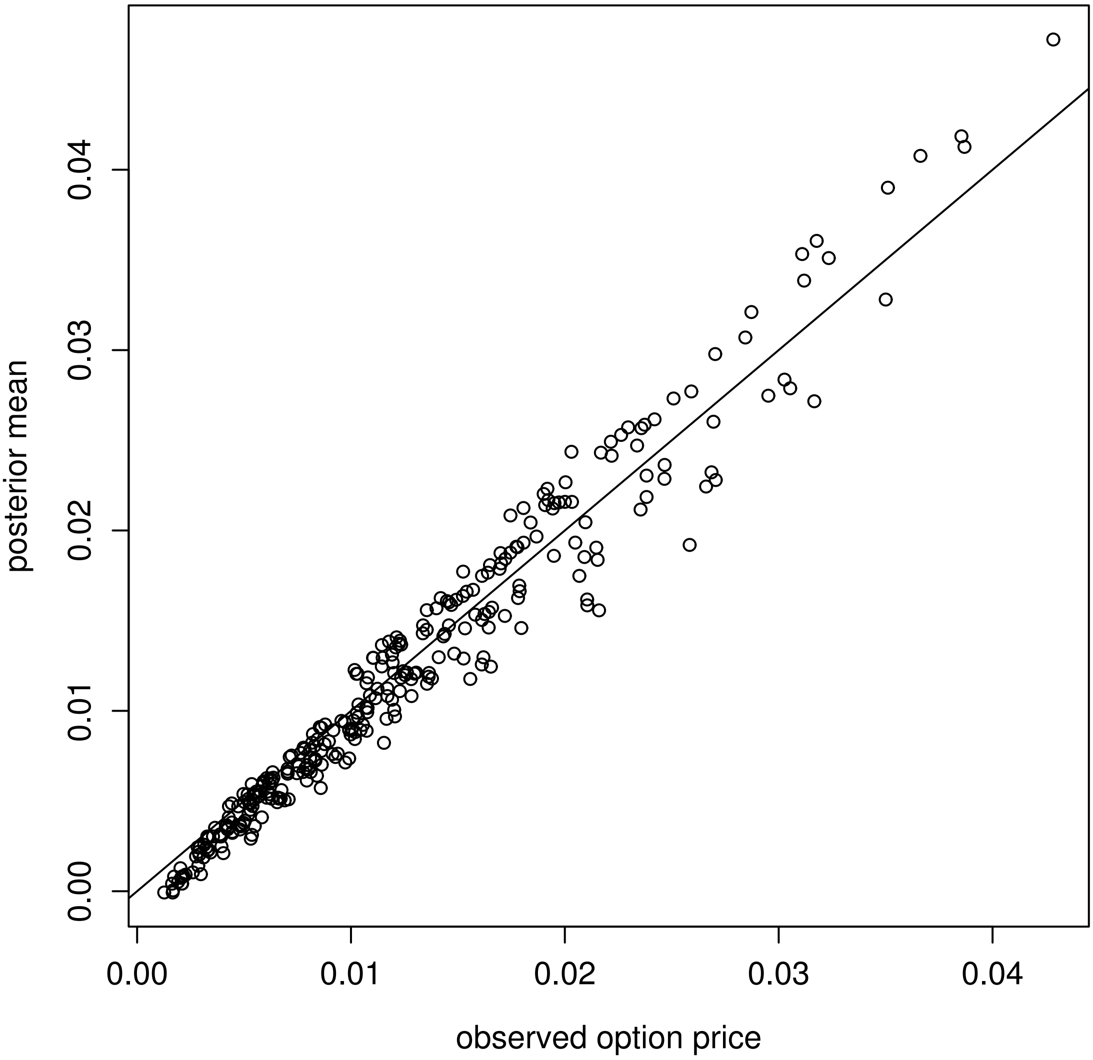

Figure 11: Scatter plot of observed option prices again option prices implied by the model.

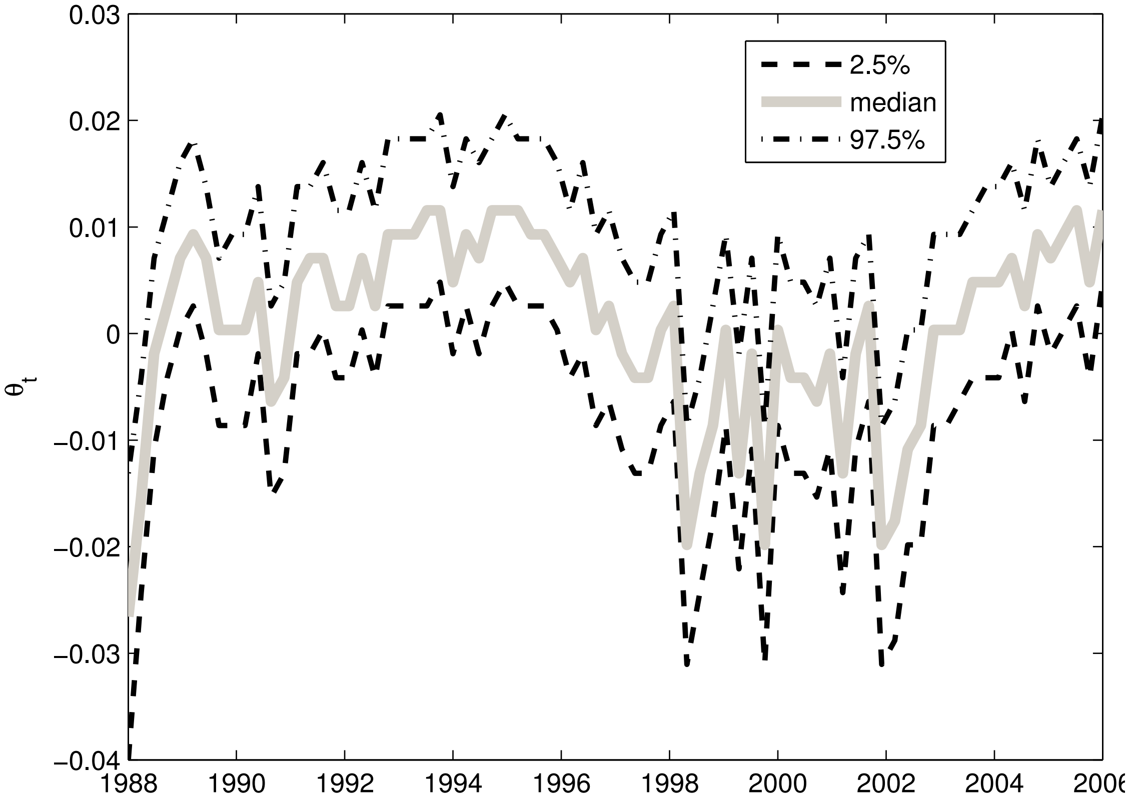

Figure 12: Time-series plot of the 2.5-97.5% confidence interval of the state θ_t inferred from put option prices.