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Abstract

We consider an economy populated by smooth ambiguity-averse agents with complete markets of securities contingent to economic scenarios, where bankruptcy is permitted but there is a penalty for it. We show that if agents' posterior belief reductions given by their "average probabilistic beliefs" do not become homogeneous then an equilibrium does not exist. It is worth noting that our main result does not imply any convergence of ambiguity perception or even the attitudes towards it. In this way, complete markets with default and punishment allows for ambiguity aversion in the long run, and the agents can disagree on their ambiguity perception but they must agree on their expected beliefs.

Keywords: Convergence of Expectations, Ambiguity Aversion. JEL Classification: D53, D81, D84.

1 Introduction

When markets are complete, one necessary condition for existence of equilibrium, in a stochastic framework with expected utility agents, is that beliefs must be locally equivalent, *i.e.*, all agent's beliefs assign null probability over the same *finite-time* events (see Harrison and Kreps (1979)). In general, equilibrium existence in infinite horizon economies is not precluded by the lack of equivalence as we can see from the conditions provided by Bewley (1972).[1] On the other

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[1]See Riedel (2003) for a class of two agents economies with von Neumann-Morgenstern utilities with heterogeneous but locally equivalent beliefs. This is a special case of additively separable preferences in which the equilibrium existence result provided by Dana (1993) holds.

hand, Araujo and Sandroni (1999) have showed that if bankruptcy is permitted, with a penalty for it, then equivalence of beliefs is a necessary condition for equilibrium existence. In turn, according to Blackwell and Dubins (1962), such equivalence implies convergence of posteriors. In other words, when agents may become bankrupt but there is a penalty for doing so, equilibrium existence implies convergence to homogeneous expectations in the long run.

An important assumption made in afore cited work is that all agents obey the expected utility hypothesis. Under this perspective, while the agents may associate subjective likelihoods to events instead of objective probabilities, the beliefs should be given by subjective probabilities. However, from the past decade the possibility that agents may not hold a single belief on scenarios has widely recognized in macroeconomics and finance. This perspective has been a classical issue in decision theory, and it goes back to the seminal work of Ellsberg (1961) proposing the notion of ambiguity by showing that may be no single probabilistic belief on the states of nature that rationalizes the pattern of behavior revealed in the well-known Ellsberg Paradox.

To model individual preferences with a negative attitude towards ambiguity, we assume the smooth ambiguity aversion model proposed by Klibanoff et al. (2005), KMM henceforth. They provided an axiomatic foundation (see also Seo (2009)) for a preference representation that allows for a separation of the perception of ambiguity and attitudes towards it. As in many applications, we assume that the smooth representation is determined by a triple (u, ϕ, μ). The attitudes towards risk is captured by a concave von Neumann-Morgenstern utility index $u : \mathbb{R}_+ \to \mathbb{R}$ that satisfies also some mild classical conditions. The attitudes towards ambiguity are described by the properties of $\phi : \mathbb{R} \to \mathbb{R}$, which we assume concave too. The second-order belief μ is defined over measurable sets of first-order probabilities and its support captures the perceived ambiguity. For a given random payoff $x : \Omega \to \mathbb{R}$, we assume in our application that the smooth representation takes the following form

$$\int_{\Delta(\Omega)} \phi\left(\int_{\Omega} v\left(x\left(\omega\right)\right) P\left(d\omega\right)\right) \mu\left(dP\right) \equiv E_{\mu}\left[\phi\left(E_P\left[v\left(x\right)\right]\right)\right],$$

where $v(x(\omega)) = u(x(\omega))$ if $x(\omega) \geq 0$ and, otherwise, the agent receives a penalty for each unit of payoff that she is short, which is captured by a constant $M > 0$ so that $v(x(\omega)) = -Mx(\omega)$ if $x(\omega) < 0$.

From the second-order belief we can derive the so called reduction or weighted belief $\mathbb{P}$, which is defined by $\mathbb{P}(A) = \int_{\Delta(\Omega)} P(A)\,\mu(dP) = E_{\mu}[P(A)]$, for any event $A \subset \Omega$. This concept is the key element for our main result as we will comment bellow.

We model an economy populated by smooth ambiguity-averse agents with complete markets of securities contingent to economic scenarios. Following Araujo and Sandroni (1999), we allow for bankruptcy but there is a penalty for it. Our main result shows that if agents' posterior reduction $\mathbb{P}^i$ are not equivalent then an equilibrium does not exist. It is worth noting that our main result does not imply any convergence of ambiguity perception or even the attitudes towards it. Accordingly, in the long run any agent should predict the *expected belief* as the same as others even they disagree in their perceived ambiguity concerning the true distribution. An important consequence of our main result is that all speculative trade associated with weighted belief heterogeneity will disappear. From this perspective, our result concerning an infinite horizon economy has the same flavor of the conclusion provided by Rigotti et al. (2008) showing that, for a finite economy with smooth agents and without aggregate uncertainty, absence of purely speculative trade means that the weighted belief have to coincide.[2]

To conclude, another interesting result that we find in our analysis says that, under our conditions imposed for the existence of equilibrium, all agents in an economy *a la* Araujo and Sandroni (1999) but with smooth ambiguity averse agents will survive.[3] Indeed, we show that there is a personal positive lower bound for the consumption in equilibrium of each agent, which is uniform over all paths. An important aspect is that we do not study the question about the survival of ambiguity averse agents in the presence of expected utility maximizers with correct beliefs, which is considered in the next section about the literature.

Related Literature

Our present work seeks to contribute to the literature on ambiguity aversion and its interplay with the theory of complete markets with default and punishment in the lines proposed by Araujo and Sandroni (1999). To the best of our knowledge, this is the first attempt to study the evolution of ambiguous beliefs as a necessary condition for the equilibrium existence in a complete market economy *a la* Araujo and Sandroni populated by smooth ambiguity averse agents. In another direction, Carvajal and Riascos (2008) analyzed the consequences of dropping completeness in the Araujo and Sadroni's model and find that if markets are sufficiently incomplete then equilibrium with trade allowing for beliefs disagreement on null events generically exists.

Since Alchian (1950) and Friedman (1953), a related problem to convergence of expectations aims to understand whether agents who do not predict future events as accurately as others are driven out of the market. A positive answer to this issue leads to the so called *market selection hypothesis* by stating that markets selects traders with more accurate beliefs. A confirmation of the market selection hypothesis has been viewed as the key arguments supporting the strong uniformity on the agents' beliefs that is imposed by the rational expectations hypothesis. In general equilibrium models with complete markets, Sandroni (2000) and Blume and Easley (2006) have showed under some regularity conditions that the market selection hypothesis holds true. One of the main message in these works is that for equally patience investors only those

[2] Rigotti et al. (2008) showed that their proposed notion of subjective beliefs (across constant acts) contain all of the information needed to predict the presence or absence of purely speculative trade. In the case of smooth ambiguity averse agents, the set of subjective beliefs contains only the prior given by the reduction $\mathbb{P}$.

[3] See our Lemma 9

with true belief or those whose forecasts merge with the true likelihoods survive. Blume and Easley (2006) also have showed that for competitive economies with incomplete markets the market selection hypothesis may fails. In another direction, Massari (2014) studies a general equilibrium model with a continuum of traders *a la* Aumann (1965) and concludes, contrary to results for economies with only finitely many traders, that risk attitudes matters for survival and that markets might select traders that does not holds accurate beliefs, but asymptotic equilibrium prices reflect accurate beliefs as claimed by the Friedman's conjecture.

More recently, Condie (2008), da Silva (2011), and Guerdjikova and Sciubba (2015) analyzed survival of ambiguity averse agents in the presence of expected utility maximizers with correct beliefs.[4] They follow the same economic framework as in Sandroni (2000) where bankruptcy is not allowed. Condie (2008) studies the case of maxmin expected utility agents as axiomatized by Gilboa and Schmeidler (1989). Condie concludes that, in general, ambiguity averse agents have no importance in the long run under the presence of aggregate risk. Da Silva (2011) considers the same framework but with more general case of agents with variational preferences (introduced by Maccheroni et al. (2006)) and states that survival of ambiguity averse agents depends on the relative "weight" assigned to the true probability model and the level of aggregate risk. Guerdjikova and Sciubba (2015) deal with smooth model agents like us but assume the presence of an objective ambiguity captured by a finite set of probability measures, uniformly perceived by all ambiguity averse agents in the market. They find special conditions on the ambiguity attitudes (including decreasing absolute ambiguity aversion) and aggregate risk in which an smooth ambiguity averse agent may survive and also determine prices in the limit. But for the case of increasing or constant absolute ambiguity aversion, smooth ambiguity averse agents only survive under no-aggregate risk and always such agents have no impact in the limit prices.

2 Notation and Framework

Consider a dynamic model with discrete time $\mathcal{T} = \{0, 1, \dots\}$. There is a finite set of agents $\mathcal{I} = \{1, \dots, I\}$, which have common information modeled by a filtered space $(\Omega, (\mathcal{F}_t)_{t\in\mathcal{T}})$, where $\Omega := \{\omega_0\} \times \prod_{t\geq 1} \mathcal{S}_t$, with ω_0 the sure state occurring at the first time and $\mathcal{S}_t = \{1, \dots, S_t\}$ the set of possible states occurring at each time $t \geq 1$. A representative element of Ω will be denoted by $\omega = (\omega_0, \omega_1, \dots)$ and $\omega^t = (\omega_0, \dots, \omega_t) \in \Omega^t := \{\omega_0\} \times \prod_{\tau=1}^{t} \mathcal{S}_\tau$. The σ-algebra generated by $(t+1)$-dimensional cylinders is $\mathcal{F}_t = \sigma(\{G_t(\omega); \omega \in \Omega\})$, where $G_t(\omega) := \{\omega^t\} \times \prod_{\tau>t} \mathcal{S}_\tau$.

Let $\mathcal{F}^0 = \cup_{t\in\mathcal{T}} \mathcal{F}_t$ be the algebra of finite-time events and $\mathcal{F} = \sigma(\mathcal{F}^0)$ the σ-algebra generated by $\mathcal{F}^0$. The filtered space $(\Omega, (\mathcal{F}_t)_{t\in\mathcal{T}}, \mathcal{F})$ represents the

[4] Easley and Yang (2014) studied the problem of survival by considering the class of loss -averse decision makers in the presence of Epstein and Zin 's (1989) preferences who are not loss-averse.

informational process known by agents. The process ω_t might be governed by a probability $\mathbb{Q}(\cdot|\omega^{t-1})$ on $\mathcal{S}_t$, which can be understood as the conditional probability given ω^{t-1} in the past. These probabilities generate the law $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ by constructing the partials $\mathbb{Q}(\omega^t) = \mathbb{Q}(\omega^{t-1})\mathbb{Q}(\omega_t|\omega^{t-1})$ on $\mathcal{F}_t$ for each $t \in \mathcal{T}$, and evoking the Kolmogorov's extension theorem (see, for instance, Shiryayev (1984)).

The set of all probabilities on a measurable space $(\Omega, \mathcal{F})$ is denoted by $\Delta(\Omega, \mathcal{F})$, or $\Delta(\Omega)$ for simplicity. If $P \in \Delta(\Omega)$, P_t denotes its restriction to $\mathcal{F}_t$, and $P_{t+1}(s|\omega^t) := \frac{P_{t+1}(\omega^t, s)}{P_t(\omega^t)}$ denotes the conditional one step ahead probability from P. Note that we can consider $P_{t+1}(\cdot|\omega^t) \in \Delta(G_t(\omega), \mathcal{F}_{t+1})$.

Given two probabilities, P and Q, we say that Q is absolutely continuous with respect to P if for $A \in \mathcal{F}$, $P(A) = 0$ implies $Q(A) = 0$, and we denote $Q \ll P$. If $Q \ll P$ and $P \ll Q$ we say that P and Q are equivalents and denote $Q \leftrightarrow P$. Again, for P and Q we denote the total variation distance between P and Q by $\|P - Q\| := \sup_{A \in \mathcal{F}} |P(A) - Q(A)|$.

The acts considered by the agents must be based on the common knowledge about the world. In this way, the possible consequences of an act at a period t will be contingent to events on $\mathcal{F}_t$. The individual choice space is a subset of

$$X = \left\{ (x_t)_{t \in \mathcal{T}};\ x_t : \Omega \to \mathbb{R} \text{ is } \mathcal{F}_t\text{-adapted and } \sup_{t,\omega} |x_t(\omega)| < \infty \right\},$$

whose dual, which contains the prices, is

$$X^* = \left\{ (p_t)_{t \in \mathcal{T}};\ p_t : \Omega \to \mathbb{R} \text{ is } \mathcal{F}_t\text{-adapted and } \sum_{t,\omega} |p_t(\omega)| < \infty \right\},$$

considering the dual pairing

$$\langle x, p \rangle = \sum_{t,\omega} x_t(\omega) p_t(\omega),$$

which generates the Mackey topology $\tau(X, X^*)$ on X, and also the weak topology $\sigma(X^*, X)$ on X^*. It is worth noting that X can be identified with the set given by

$$\left\{ x : \bigcup_{t \in \mathcal{T}} (\{t\} \times \Omega^t) \to \mathbb{R};\ \sup_{t,\omega} |x(t, \omega^t)| < \infty \right\},$$

which in turn is basically ℓ^∞ since $\bigcup_{t \in \mathcal{T}} (\{t\} \times \Omega^t)$ is a enumerable set.

3 Market, Agents and Optimal Choices

At period 0 agent $i \in \mathcal{I}$ can trade contingent claims for all periods, in other words, each agent i choose an asset allocation $k^i = (k^i_t)_{t \geq 1} \in X$, where $k^i_t(\omega)$

represents the amount that i will receive (deliver in the negative case) at time t if ω^t occurs. Agent i has a positive consumption at period 0 such that

$$c_0^i = e_0^i - \langle q, k^i \rangle, \tag{1}$$

where $q \in X_+^*$ is the price of assets. Each agent is endowed with an initial consumption stream $e^i \in X_+$ satisfying, for all $i \in \mathcal{I}$

$$\underline{e} < e^i < \sum_j e^j < \bar{e},$$

for positive constants $\underline{e}$ and $\bar{e}$. For $t \geq 1$, agent i's consumption stream derived of his choice k^i is $c_t^i = (e_t^i + k_t^i)^+$, and $d_t^i = (e_t^i + k_t^i)^-$ is the amount which he is short of (we also denote $k_0^i := -\langle q, k^i \rangle$ in order to get $c_0^i = e_0^i + k_0^i \geq 0$). We assume the existence of a penalty for each unity of d_t^i, which is constant and denoted by $M^i > 0$.

As discussed in our Introduction, we do not rule out negative attitudes towards ambiguity implicitly done by Araujo and Sandroni (1999). Actually, we allow for ambiguity-sensitive behavior by considering that each agent i behaves in a way inspired by the smooth ambiguity averse model proposed by Klibanoff et al. (2005). Following Araujo and Sandroni (1999), each agent i is endowed with a utility index v_i over $\mathbb{R}$ determined by u_i and M^i in the following way:

$$v_i(x) = \begin{cases} u_i(x) & \text{if} \quad x > 0 \\ -M^i x & \text{if} \quad x \leq 0 \end{cases}.$$

The second order belief of agent i is denoted by μ_i and the corresponding support $Supp\left(\mu^i\right)$ captures the ambiguity perception of the agent.[5] For each first order probability $P \in Supp\left(\mu^i\right)$ we have the discounted expected utility associated to each profile $\left(c_0^i, k^i\right)$ given by

$$\mathbb{E}_P \left[\sum_{t=0}^{\infty} \beta^t v_i(e_t^i + k_t^i) \right].$$

Each agent i is also endowed with a function ϕ_i which captures her ambiguity attitudes. To summarize, the way in which each agent i ranks alternatives $c^i \equiv \left(c_0^i, k^i\right) \in \mathbb{R}_+ \times X$ is determined by the couple $\left(u_i, \phi_i, \mu^i, e^i, M^i\right)$ that generates the functional

$$V^i(c^i) \equiv V^i(c_0^i, k^i) = \mathbb{E}_{\mu^i} \left\{ \phi_i \left(\mathbb{E}_P \left[\sum_{t \geq 0}^{\infty} \beta^t v_i(e_t^i + k_t^i) \right] \right) \right\}. \tag{2}$$

From the second-order belief μ_i we can derive the so called reduction or weighted belief $\mathbb{P}^i$ of the agent i:

$$\mathbb{P}^i\left(A\right) := E_{\mu^i}\left[P\left(A\right)\right],$$

[5] $Supp\left(\mu^i\right)$ is the smaller closed set (w.r.t. the vague topology) in Δ whose complement has μ^i null measure.

for any event $A \subset \Omega$. Also, we assume that the utility index u_i and the ambiguity index ϕ_i are twice continuously differentiable with

$$u_i(0) = 0, u_i', \phi_i' > 0, u_i'' < 0, \phi_i'' \leq 0 \text{ and } \lim_{x \to 0^+} u_i(x) = \infty.$$

Definition 1 *A competitive equilibrium with penalties is described by a profile of allocations and price* $((\bar{c}_0^i, \bar{k}^i)_{i \in \mathcal{I}}, \bar{q})$ *such that each agent* i *optimizes and markets clear, i.e.*

$$\sum_i c^i = \sum_i e^i,$$

$$\sum_i k^i = 0.$$

Lemma 2 *In every equilibrium with penalties, agent* i*'s first order conditions*

$$\beta^t u_i'(c_t^i(\omega^t)) \mathbb{E}_{\mu^i} \{\phi_i'(\mathbb{E}_P[U_i(c^i)]) P(\omega^t)\} = \lambda(i) q_t(\omega^t) \tag{3}$$

are satisfied, where $U_i(c^i) := \sum_t \beta^t u_i(c_t^i)$, $\omega^t \in \Omega^t$ *and* $\lambda(i) > 0$ *is agent* i*'s Lagrange multiplier.*

4 Ambiguity in the Long Run

We show in this section our main result which says that a necessary condition for existence of equilibrium is the equivalence between the reductions or expected beliefs of every individual. Moreover, by Blackwell and Dubins (1962), the expected beliefs must eventually become homogeneous. Araujo and Sandroni (1999) assumed that individuals are risk averse, and further features on attitude toward risk plays no role. In our case, the level of risk aversion still not important and the type of non-positive attitude toward ambiguity also does not matter.

Before presenting our result about global equivalence of the expected beliefs, we need to introduce an important regularity condition over the agents perceived ambiguity.

Definition 3 *We say that the agent* i *obeys the* unambiguous convergence to null events *property if for any sequence of events* $\{A_t\}_{t \geq 1}$, *where* $A_t \in \mathcal{F}_t$, $\forall t \geq 1$, *if*

$$\lim_{t \to \infty} \mathbb{P}^i(A_t) = 0,$$

then

$$\lim_{t \to \infty} P(A_t) = 0,$$

uniformly for $P \in Supp(\mu^i)$.

As usual, uniform convergence is a kind of strong condition on the mode of convergence. In our case, given an agent i, by imposing the unambiguous convergence to null events property we mean that if the expected belief $\mathbb{P}^i$ says that a sequence of events converge to a miracle then all first-order priors describing the perceived ambiguity must agree with this convergence and also it must be uniform over all priors in $Supp\left(\mu^i\right)$. We note that this condition is automatically satisfied when we assume finite supports $Supp\left(\mu^i\right) = \left\{P_1^i, ..., P_k^i\right\} \subset \Delta\left(\Omega\right)$.

Theorem 4 *Suppose that all agents in the economy satisfies the unambiguous convergence to null events property. Let* $((\bar{c}_0^i, \bar{k}^i)_{i\in\mathcal{I}}, \bar{q})$ *be an equilibrium with penalties, then* $\mathbb{P}^i \ll \mathbb{P}^j$ $\forall i, j \in \mathcal{I}$. *Moreover,*

$$\lim_{t\to\infty} \|\mathbb{P}^i(\cdot|\omega^t) - \mathbb{P}^j(\cdot|\omega^t)\| = 0,\ \mathbb{P}^i - a.s., \text{ for all } i, j \in \mathcal{I}.$$

5 Existence of Equilibrium

In the previous section we have considered an economy *a la* Araujo and Sandroni (1999) but with smooth ambiguity averse agents holding second-order beliefs that satisfy the "unambiguous convergence to null events" condition. Our main result says that a necessary condition for the existence of a competitive equilibrium without default is that agent's expected beliefs are globally equivalent. It is clear that in order to get that this result is economically meaningful we need to give reasonable conditions in which the equilibrium exists. The following condition proposed by Sandroni (1994) will be used in our next result about the existence of a competitive equilibrium.

Definition 5 *We say that a set of probabilities* $\mathcal{P}$ *displays strong compatibility condition if there is a constant* $K > 0$ *such that*

$$\mathbb{P}_1(A) \leq K\mathbb{P}_2(A),\ \forall A \in \mathcal{F}$$

for any $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}$.

We note that this condition can be rewritten by saying that for all $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}$

$$\sup_{A\in\mathcal{F}} \frac{\mathbb{P}_1(A)}{\mathbb{P}_2(A)} < +\infty.$$

Under the strong compatibility condition, the next result finds that there exist a profile of default penalties that ensures the existence of equilibrium with penalties.[6] Concerning the interpretation of default penalties, Dubey et al. (2005) claims that it might be interpreted in terms of some extra-economic punishments (prison or a self blame recognized generating utility loss). On the other hand, Zame (1993) come up with a interpretation of default penalties based on the notion of a non-modeled economic punishment given, for instance, by reputation loss or exclusion from credit markets.

[6] In the proof of Proposition 6we provide an explicit lower bound for each individual default penalty.

Proposition 6 *Suppose that* $\{\mathbb{P}^i\}_{i\in\mathcal{I}}$ *displays the strong compatibility condition, then there exists a profile* $(M_i)_{i\in\mathcal{I}}$ *of penalties such that there exists a competitive equilibrium with penalties.*

Let us examine some examples fitting the conditions of Proposition 6. First, consider the case generated by a finite set $\mathcal{S}_t = \{s_1, s_2\}$ for every $t \geq 1$. Assume also that there are two probability measures P and Q over Ω generated by i.i.d. trials that assign $p(s_1)$ and $q(s_1)$ to state 1, respectively, such that $0 < p(s_1) < q(s_1) < 1$. Also, we assume that the economy is populated by two agents second order beliefs μ_1 and μ_2 such that $Supp(\mu_1) = Supp(\mu_2) = \{P, Q\}$ and $\mu_1(P) = m_1 < m_2 = \mu_2(P)$. We note that for each partial history $\omega^t = (\omega_0, \ldots, \omega_t)$

$$\begin{aligned}\frac{\mathbb{P}_2(\omega^t)}{\mathbb{P}_1(\omega^t)} &= \frac{m_2 \prod_{s=0}^t p(\omega_s) + (1-m_2)\prod_{s=0}^t q(\omega_s)}{m_1 \prod_{s=0}^t p(\omega_s) + (1-m_1)\prod_{s=0}^t q(\omega_s)} \\ &= \frac{\prod_{s=0}^t q(\omega_s) + m_2\left(\prod_{s=0}^t p(\omega_s) - \prod_{s=0}^t q(\omega_s)\right)}{\prod_{s=0}^t q(\omega_s) + m_1\left(\prod_{s=0}^t p(\omega_s) - \prod_{s=0}^t q(\omega_s)\right)} \\ &= \frac{1 + m_2\left(\prod_{s=0}^t \frac{p(\omega_s)}{q(\omega_s)} - 1\right)}{1 + m_1\left(\prod_{s=0}^t \frac{p(\omega_s)}{q(\omega_s)} - 1\right)}\end{aligned}$$

We have that $f(x) = \frac{1+m_2 x}{1+m_1 x}$ with $x \in [-1, \infty)$ is a increasing and continuos function such that $\frac{1-m_2}{1-m_1} \leq f(x) \leq \frac{m_2}{m_1}$, and then $\frac{1-m_2}{1-m_1} \leq \frac{\mathbb{P}_2(\omega^t)}{\mathbb{P}_1(\omega^t)} \leq \frac{m_2}{m_1}$.

Now, in another example, assume that P and Q are two arbitrary probabilities over Ω and define $[P, Q] := \{\alpha P + (1-\alpha)Q : \alpha \in [0,1]\}$. Also, assume that $\mathcal{I} = \{1, 2\}$ and $\mathbb{P}_i(A) = \int_0^1 f_i(\alpha) P_\alpha d\alpha$, where $P_\alpha = \alpha P + (1-\alpha)Q$ and $f_i : [0,1] \to [0, \infty)$ is a continuous function satisfying $\int_0^1 f_i(\alpha)d\alpha = 1$. Thus, the support of μ_i is given by the convex set $[P, Q]$. Provide that $\frac{1}{K} \leq \frac{f_1(x)}{f_2(x)} \leq K$ for some $K > 0$, the reductions $\mathbb{P}_1$ and $\mathbb{P}_2$ are strongly compatible.

Appendix

Proof of Lemma 2:
Proof. In an equilibrium with penalties there is no default and by the assumption given in the Inada conditions the equilibrium allocations must be positive. So, since we have interior solutions of maximization problems we get first order conditions (see, for instance Luenberger (1969) section 9.3, Theorem 1). ■

Before the proof of our Theorem **4**, we need the following lemma:

Lemma 7 *Let* $P, Q \in \Delta$*, if* P *is not absolutely continuous with respect to* Q *then there exists a sequence* $A_t \in \mathcal{F}_t$ *such that* $Q(A_t) \to 0$ *and* $P(A_t) > \delta > 0$ $\forall t \in \mathcal{T}$*.*

Proof. By hypothesis there exists $A \in \mathcal{F}$ such that $Q(A) = 0$ but $P(A) > \delta$ for some $\delta \in (0, 1)$. Since $\mathcal{F} = \sigma(\cup_t \mathcal{F}_t)$ and $\cup_t \mathcal{F}_t$ is an algebra, by Carathéodory extension (see Shiryayev (1984)) $Q(A) = \inf\{Q(B); A \subset B \in \cup_t \mathcal{F}_t\}$. So, for each $n \in \mathbb{N}$, $\exists A_{t_n} \in \mathcal{F}_{t_n}$ such that $A \subset A_{t_n}$ and $Q(A_{t_n}) < \frac{1}{n}$, with $t_n < t_{n+1}$ because $\mathcal{F}_t \subset \mathcal{F}_{t+1}$. On the other hand $P(A_{t_n}) \geq P(A) > \delta$. For $t \in (t_n, t_{n+1})$ put $A_t = A_{t_n}$. ■

Proof of Theorem 4:

Proof. Assume that $\mathbb{P}^i$ is not absolutely continuous with respect to $\mathbb{P}^j$ for some $i, j \in \mathcal{I}$. Thus, by Lemma 7, there exists a sequence of events $A_t \in \mathcal{F}_t$ and $\delta > 0$ such that $\lim_t \mathbb{P}^j(A_t) = 0$ but $\mathbb{P}^i(A_t) \geq \delta$ for all $t \geq 1$. By the first order condition as obtained in our Lemma 2 we get

$$\begin{aligned}\left\langle q_t, \frac{1}{\beta^t}\chi_{A_t}\right\rangle &= \frac{1}{\lambda_i}\mathbb{E}_{\mu^i}\left\{\phi_i'\left(\mathbb{E}_P\left[U_i\left(c^i\right)\right]\right)\mathbb{E}_P\left[u_i'\left(\bar{c}^i\right)\chi_{A_t}\right]\right\} \\ &\geq \frac{1}{\lambda_i}\phi_i'\left(U_i\left(\bar{e}\right)\right)u_i'\left(\bar{e}\right)\mathbb{P}^i\left(A_t\right) \\ &\geq \frac{1}{\lambda_i}\phi_i'\left(U_i\left(\bar{e}\right)\right)u_i'\left(\bar{e}\right)\delta =: \eta > 0.\end{aligned}$$

Since the unambiguous convergence to null events property holds for all agents, we obtain that if $\lim_t \mathbb{P}^j(A_t) = 0$ then for any $n \in \mathbb{N}$ there is $t \geq 1$ such that $P(A_t) < \frac{1}{n}$ for all $P \in Supp\left(\mu^j\right)$. So, let $t_1 \geq 1$ such that for any $P \in Supp\left(\mu^j\right)$ we get

$$P\left(A_{t_1}\right) < \frac{K}{M_j + u_j\left(\bar{e}\right)},$$

where $K := u_j\left(\bar{c}_0^j + \eta\right) - u_j\left(\bar{c}_0^j\right) > 0$. Define $\widehat{c}^j$ and $\widehat{k}^j$ by

$$\widehat{c}_0^j := \bar{c}_0^j + \eta,\ \widehat{k}_t^j = \bar{k}_t^j \text{ if } t \neq t_1, \text{ and } \widehat{k}_{t_1}^j = \bar{k}_{t_1}^j - \frac{\chi_{A_{t_1}}}{\beta^{t_1}},$$

and note that

$$\widehat{c}_0^j - e_0^j + \left\langle q, \widehat{k}^j\right\rangle = \bar{c}_0^j + \eta - e_0^j + \left\langle q, \widehat{k}^j\right\rangle - \left\langle q_{t_1}, \frac{\chi_{A_{t_1}}}{\beta^{t_1}}\right\rangle \leq 0,$$

increasing $\widehat{c}_0^j$ if we need to. Now, we can verify that $\left(\widehat{c}_0^j, \widehat{k}^j\right)$ is a better choice than $(\bar{c}_0^j, \bar{k}^j)$ for the agent j. Actually,

$$\begin{aligned}V^j(\widehat{c}^j) &= \mathbb{E}_{\mu^j}\left\{\phi_j\left(\mathbb{E}_P\left[\sum \beta^t v_j(\hat{c}_t^j)\right]\right)\right\} \\ &= \mathbb{E}_{\mu^j}\left\{\phi_j\left(\mathbb{E}_P\left[\sum \beta^t v_j(\bar{c}_t^j) + (v_j(\hat{c}_0^j) - v_j(\bar{c}_0^j)) + \beta^{t_1}(v_j(\hat{c}_{t_1}^j) - v_j(\bar{c}_{t_1}^j))\right]\right)\right\} \\ &\geq \mathbb{E}_{\mu^j}\left\{\phi_j\left(\mathbb{E}_P\left[\sum \beta^t v_j(\bar{c}_t^j) + K - \chi_{A_{t_1}}\beta^{t_1}(\beta^{-t_1}M_j + u_j(\bar{e}))\right]\right)\right\} \\ &> \mathbb{E}_{\mu^j}\left\{\phi_j\left(\mathbb{E}_P\left[\sum \beta^t v_j(\bar{c}_t^j)\right] + K - P(A_{t_1})(M_j + u_j(\bar{e}))\right)\right\} \\ &> V^j(\bar{c}^j).\end{aligned}$$

For the second part, the proof is an immediate consequence of the Blackwell and Dubins (1962)'s Theorem. ■

Next lemmas provide important results that will be used in the proof of our Proposition 6 about equilibrium existence.

Lemma 8 *If $W : X \to \mathbb{R}$ is an increasing and concave function, then for each $\bar{p} \in X^*$, the mapping*

$$\psi(r) := \max\{W(c) : \langle \bar{p}, c\rangle \leq r, c \geq 0\}$$

is an increasing and concave real function.

Proof. If $r < r'$, then $\{W(c) : \langle \bar{p}, c\rangle \leq r, c \geq 0\} \subset \{W(c) : \langle \bar{p}, c\rangle \leq r', c \geq 0\}$, and so $\psi(r) \leq \psi(r')$. If $r, r' > 0$ and $\alpha \in [0,1]$, by concavity of W

$$W(\alpha c + (1-\alpha)c') \geq \alpha W(c) + (1-\alpha)W(c'),$$

therefore

$$\begin{aligned}
&\max\{W(\alpha c + (1-\alpha)c') : \langle \bar{p}, c\rangle \leq r, \langle \bar{p}, c'\rangle \leq r' \text{ and } c, c' \geq 0\}\\
&\geq \max\{\alpha W(c) + (1-\alpha)W(c') : \langle \bar{p}, c\rangle \leq r, \langle \bar{p}, c'\rangle \leq r' \text{ and } c, c' \geq 0\}\\
&= \max\{\alpha(c) : \langle \bar{p}, c\rangle \leq r, c \geq 0\} + \max\{(1-\alpha)W(c') : \langle \bar{p}, c'\rangle \leq r', c' \geq 0\}\\
&= \alpha\psi(r) + (1-\alpha)\psi(r').
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\psi(\alpha r + (1-\alpha)r')\\
&= \max\{W(\alpha c + (1-\alpha)c') : \langle \bar{p}, \alpha c + (1-\alpha)c'\rangle \leq \alpha r + (1-\alpha)r' \text{ and } c, c' \geq 0\}\\
&\geq \max\{W(\alpha c + (1-\alpha)c') : \langle \bar{p}, c\rangle \leq r, \langle \bar{p}, c'\rangle \leq r' \text{ and } c, c' \geq 0\}.
\end{aligned}$$

■

Lemma 9 *If $\{\mathbb{P}^i\}_{i\in\mathcal{I}}$ displays strong compatibility condition and $(c^i)_{i\in\mathcal{I}}$ is an equilibrium, then $\exists l_i > 0$ such that $c^i_t \geq l_i$ for each $i \in \mathcal{I}$ and all $t \in \mathcal{T}$.*

Proof. Since $\underline{e} < \sum_i e^i = \sum_i c^i$, for each node ω^t there exists a $j \in \mathcal{I}$ such that $c^j_t(\omega^t) > \underline{e}$. For a fixed $i \in \mathcal{I}$, from the first order conditions (3)

$$\frac{u'_j(c^j_t(\omega^t))\mathbb{E}_{\mu^j}\{\phi'_j(\mathbb{E}_P[U_j(c^j)]P(\omega^t))\}}{u'_i(c^i_t(\omega^t))\mathbb{E}_{\mu^i}\{\phi'_i(\mathbb{E}_P[U_j(c^i)]P(\omega^t))\}} = \frac{\lambda_j}{\lambda_i},$$

so

$$\frac{u'_j(\underline{e})}{u'_i(c^i_t(\omega^t))}\frac{\phi'_j(U_j(\underline{e}))K\mathbb{P}_i(\omega^t)}{\phi'_i(U_i(\bar{e}))\mathbb{P}_i(\omega^t)} \geq \frac{\lambda_j}{\lambda_i},$$

therefore $u'_i(c^i_t(\omega^t)) \leq \frac{\lambda_i u'_j(\underline{e})\phi'_j(U_j(\underline{e}))K}{\lambda_j\phi'_i(U_i(\bar{e}))}$ and $c^i_t(\omega^t) \geq u'^{-1}_i\left(\frac{\lambda_i u'_j(\underline{e})\phi'_j(U_j(\underline{e}))K}{\lambda_j\phi'_i(U_i(\bar{e}))}\right)$. Finally, put $l_i = \min_{j\neq i}\left\{\underline{e}, u'^{-1}_i\left(\frac{\lambda_i u'_j(\underline{e})\phi'_j(U_j(\underline{e}))K}{\lambda_j\phi'_i(U_i(\bar{e}))}\right)\right\}$ and notice this constant does not depend on the history ω. ■

Lemma 10 *Under the same assumptions of the previous lemma, we get* $q_t(\omega^t) \leq L_i \beta^t \mathbb{P}^i(\omega^t)$*, where* L_i *is a positive constant.*

Proof. By the first order conditions and using the Lemma 9 bound we get

$$\begin{aligned} q_t(\omega^t) &= \frac{1}{\lambda(i)}\beta^t u_i'(c_t^i(\omega^t))\mathbb{E}_{\mu^i}\{\phi_i'(\mathbb{E}_P[U_i(c^i)])P(\omega^t)\} \\ &\leq \frac{1}{\lambda(i)}\beta^t u_i'(l_i)\mathbb{E}_{\mu^i}\{\phi_i'(\mathbb{E}_P[U_i(l_i)])P(\omega^t)\} \\ &= \frac{1}{\lambda(i)}\beta^t u_i'(l_i)\phi_i'(U_i(l_i))\mathbb{P}^i(\omega^t) \end{aligned}$$

then we take $L_i = \frac{1}{\lambda(i)} u_i'(l_i)\phi_i'(U_i(l_i))$. ∎

Before the next Lemma, note that X can be viewed as a l_∞ space. Since $x \in X$ is $\mathcal{F}_t$-adapted it is a function on the nodes

$$\mathcal{N} = \{(t, \omega^t)/t \in \{0, 1, 2, \dots\},\ \omega^t \in \Omega^t\} = \cup_{t=0}^{\infty}(\{t\} \times \Omega^t)$$

which is an enumerable set.

Lemma 11 *The utility function*

$$\tilde{V}^i(c) = \int_{\Delta(\Omega)} \phi_i \left(\int_{\Omega} U_i(c(\omega))P(d\omega) \right) \mu^i(dP)$$

is Mackey continuous in X*.*

Proof. Remember $\tilde{V}^i$ is a concave function, so $-\tilde{V}^i$ is convex. The Mackey continuity follows from Proposition 2 section 3.3.2 in Aubin (1982), because $-\tilde{V}^i \leq 0$. ∎

Proof of Proposition 6:

Proof. If we consider the utilities

$$\tilde{V}_i(c) = \mathbb{E}_{\mu^i}\left\{\phi_i\left(\mathbb{E}_P\left[U_i(c)\right]\right)\right\},$$

by Lemma 11, Bewley (1972) gives that there exists $((\bar{c}^i)_{i\in\mathcal{I}}, \bar{p}) \in X_+^I \times X_+^*$ such that:

$$\sum_i (\bar{c}^i - e^i) = 0;$$

and,

$$\bar{c}^i = \arg\max\left\{\tilde{V}(c^i) : \bar{p}(c^i - e^i) \leq 0,\ c^i \geq 0\right\}.$$

Define $\bar{k}^i$ by $\bar{k}_0^i := 0$, $\bar{k}_t^i := \bar{c}_t^i - \bar{e}_t^i$, $t \geq 1$ and $\bar{q}$ by $\bar{q}_0 := 0$, $\bar{q}_t := \frac{1}{p_0}\bar{p}_t$ $t \geq 1$. We claim that $((\bar{c}^i, \bar{k}^i)_{i\in\mathcal{I}}, \bar{q})$ is equilibrium with penalty for suitable $(M_i)_{i\in\mathcal{I}}$. Note that $(\bar{c}^i, \bar{k}^i)$ is in the constraint (1) since by $\langle \bar{p}, \bar{c}^i - e^i\rangle = 0$ we get $\bar{c}_0^i - e_0^i + \langle \bar{q}, \bar{k}^i\rangle = 0$, furthermore $\sum_i \bar{k}^i = \sum_i (\bar{c}^i - e^i)$. Now, consider an arbitrary (c^i, k^i) satisfying the constraint (1). As $c_t^i - d_t^i = e_t^i + k_t^i$, multiplying both sides by $\bar{p}_t$ and summing over $t > 0$ we get

$$\sum_{t>0} \bar{p}_t c_t^i - \sum_{t>0} \bar{p}_t d_t^i = \sum_{t>0} \bar{p}_t e_t^i + \bar{p}_0 \sum_{t>0} \bar{q}_t k_t^i,$$

by summing over all nodes and rearranging we get

$$\langle \bar{p}, c^i \rangle = \langle \bar{p}, e^i \rangle + \langle \bar{p}, d^i \rangle.$$

Denote by $\bar{r} = \langle \bar{p}, e^i \rangle$ and

$$\psi_i(r) = \max \left\{ \tilde{V}_i(c^i) : c^i \in X_+ \text{ and } \langle \bar{p}, c^i \rangle \leq r \right\}.$$

Since ψ_i is concave (see Lemma 8) if $r < r'$

$$\psi_i(r) - \psi_i(r') \geq -D_+\psi_i(r)(r' - r)$$

where $D_+\psi_i(r)$ denotes the derivative from the right of ψ_i at r. By the first order conditions we get that

$$\beta^t u_i'(c_t^i(\omega^t)) \mathbb{E}_{\mu^i} \left\{ \phi_i' \left(\mathbb{E}_P \left[U_i(c^i) \right] \right) P(\omega^t) \right\} = \lambda_i \bar{p}_t(\omega^t), \ \forall \omega^t \in \Omega^t,$$

so, by Lemma 10 we get $q_t(\omega^t) \leq L_i \beta^t \mathbb{P}^i(\omega^t)$, where the constant incorporates p_0. Therefore, by this bound and the concavity of ϕ_i and ψ_i

$$\begin{aligned}
V_i(\bar{c}^i) - V_i(c^i) =& \mathbb{E}_{\mu^i} \left\{ \phi_i \left(\mathbb{E}_P \left[\sum \beta^t u_i(\bar{c}^i) \right] \right) \right\} - \mathbb{E}_{\mu^i} \left\{ \phi_i \left(\mathbb{E}_P \left[\sum \beta^t (u_i(c^i) - M_i d_t^i) \right] \right) \right\} \\
&\geq \mathbb{E}_{\mu^i} \left\{ \phi_i \left(\mathbb{E}_P \left[\sum \beta^t u_i(\bar{c}^i) \right] \right) \right\} - \mathbb{E}_{\mu^i} \left\{ \phi_i \left(\mathbb{E}_P \left[\sum \beta^t u_i(c^i) \right] \right) \right\} \\
&+ \mathbb{E}_{\mu^i} \left\{ \phi_i' \left(\mathbb{E}_P \left[\sum \beta^t u_i(c^i) \right] \right) \mathbb{E}_P \left[\sum \beta^t M_i d_t^i \right] \right\} \\
&\geq \psi_i(\bar{r}) - \psi_i(\bar{r} + \langle \bar{p}, d^i \rangle) + \phi_i'(U_i(\bar{e})) M_i \mathbb{E}_{\mathbb{P}^i} \left[\sum \beta^t d_t^i \right] \\
&\geq -D_+\psi_i(\bar{r}) \langle \bar{p}, d^i \rangle + \phi_i'(U_i(\bar{e})) M_i \mathbb{E}_{\mathbb{P}^i} \left[\sum \beta^t d_t^i \right] \\
&\geq \left(\phi_i'(U_i(\bar{e})) M_i - D_+\psi_i(\bar{r}) L_i \right) \mathbb{E}_{\mathbb{P}^i} \left[\sum \beta^t d_t^i \right],
\end{aligned}$$

thus, if $M_i \geq \frac{D_+\psi_i(\bar{r}) L_i}{\phi_i'(U_i(\bar{e}))}$, $((\bar{c}^i, \bar{k}^i)_{i \in \mathcal{I}}, \bar{q})$ is an equilibrium with penalties. ■

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